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A new LES model derived from generalized Navier–Stokes equations with nonlinear viscosity

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ABSTRACT

Large Eddy Simulation (LES) is a very useful model for simulating turbulent flows (see Argyropoulos and Markatos, 2015, Guermond et al., 2004 or Sagaut, 2006, for example). One of the possible ways to derive the LES equations is to apply a filter operator to the Navier–Stokes equations, obtaining a new equation governing the behavior of the filtered velocity. This approach introduces the so called *subgrid-scale stress tensor* in the equations, that must be expressed in terms of the filtered velocity to close the problem. One of the most popular models is that proposed by Smagorinsky (1963), where the subgrid-scale stress tensor is modeled by introducing an *eddy viscosity*.

In this work, we shall propose a new approximation to this problem by applying the filter, not to the Navier–Stokes equations, but to a generalized version of these equations with nonlinear viscosity. That is, we shall introduce a nonlinear viscosity, not as a procedure to close the subgrid-scale stress tensor, but as part of the model itself (see below). Consequently, we shall need a different method to close the subgrid-scale stress tensor, and we shall use the Clark approximation, where the Taylor expansion of the subgrid-scale stress tensor is computed (see Carati et al., 2001 and Vreman et al., 1966).

1. Introduction

It is generally accepted that Navier–Stokes equations accurately model the behavior of incompressible viscous fluids on macroscopic scales. Today it is possible to simulate these equations numerically thanks to high computing power available. However, Direct Numerical Simulation (DNS) is limited to relatively low Reynolds numbers, because to simulate Navier–Stokes equations for a given value of Reynolds number (R_e), at least, $\mathcal{O}(R_e^{9/4})$ degrees of freedom are needed (see [1]).

There are several methods to overcome this problem. Reynolds Averaged Navier–Stokes (RANS) equations are a class of models extensively used to provide time-averaged solutions of the Navier–Stokes equations, for a high Reynolds number, by the scientific community (see, for example, [2–4]). Among them, the $k-\varepsilon$ models are, by far, the most widely used and tested two-equations RANS models (standard $k-\varepsilon$ model can be found in [5]). Large Eddy Simulation (LES) models are another class of models frequently used to simulate turbulent flows (interested readers can find several LES models in [2,6–8,1,9] or [10]). Usually, LES is presented as an "averaged" or "filtered" version of Navier–Stokes equations. We also want to mention the Detached-Eddy Simulation (DES) models, that were originated with the purpose of combining LES and RANS approaches (see for example [11] or [12]).

In this paper we aim to obtain a new LES model, therefore we consider a "filter" operator $f \to \bar{f}$ (a spatial filter, a time filter or both), and let us assume that it is linear and commutes with spatial and time derivatives (see [10] for different

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examples of this kind of filters). If we apply this filter to Navier–Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\nabla \mathbf{u})\,\mathbf{u} + \frac{1}{\rho_0}\nabla p - \nu\Delta\mathbf{u} = \mathbf{f}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

(where $\mathbf{u}$ is the velocity field, p is the pressure, and $\mathbf{f}$ is the acceleration due to external forces), we obtain

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\nabla \bar{\mathbf{u}})\,\bar{\mathbf{u}} + \frac{1}{\rho_0}\nabla \bar{p} - \nu\Delta\bar{\mathbf{u}} = \bar{\mathbf{f}} - \nabla\cdot\tau, \tag{3}$$

$$\nabla \cdot \bar{\mathbf{u}} = 0, \tag{4}$$

$$\tau_{ij} = \overline{u_i u_j} - \bar{u}_i\bar{u}_j, \tag{5}$$

where τ is the so called *subgrid-scale stress tensor*. To close the LES model it is necessary to express the subgrid-scale stress tensor only in terms of $\bar{\mathbf{u}}$. One of the most popular LES models is that proposed by Smagorinsky [13], where the subgrid-scale stress tensor is modeled by introducing an *eddy viscosity* ν_e such that

$$\tau = -2\nu_e\bar{\mathbf{D}}, \qquad \nu_e = (C_S\Delta x)^2\sqrt{2}\,|\bar{\mathbf{D}}|, \tag{6}$$

$$\bar{\mathbf{D}} = \frac{1}{2}\left(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T\right), \qquad |\bar{\mathbf{D}}| = \left(\sum_{i,j=1}^{3}\bar{D}_{ij}\bar{D}_{ij}\right)^{1/2}, \tag{7}$$

Δx is the subgrid-scale characteristic length and C_S is a constant chosen to allow the model to emulate the kinetic-energy dissipation predicted by Kolmogorov (see [6,1] or [9]).

The Smagorinsky model is sometimes too dissipative, and it has other theoretical problems, such as the fact that the subgrid-scale stress tensor in (6) is odd in $\bar{\mathbf{u}}$, when by definition (see (5)), it must be even in $\bar{\mathbf{u}}$. This happens because (6) does not try to approximate the subgrid-scale stress tensor as defined in (5), but (as we have said before) to emulate the kinetic-energy dissipation predicted by Kolmogorov.

2. Deriving the new LES model

In the previous section, we have pointed out that the Smagorinsky model introduces dissipation when approximating (5) by (6)–(7). In this section, we shall propose to model this dissipation by introducing a nonlinear viscosity in the Navier–Stokes equations and then we shall apply the filter. Acting in such a way we introduce the dissipation predicted by Kolmogorov directly in our model and, when applying the filter, we shall close the LES model by approximating the subgrid-scale stress tensor using Clark approximation (see Section 2.2).

2.1. Introducing the nonlinear viscosity

Let us consider the following generalization of Navier–Stokes equations

$$\rho_0\left(\frac{\partial \mathbf{u}}{\partial t} + (\nabla\mathbf{u})\,\mathbf{u}\right) = \rho_0\mathbf{f} + \nabla\cdot\mathbf{T}, \tag{8}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{9}$$

where the stress tensor $\mathbf{T}$ is given by

$$\mathbf{T} = -p\mathbf{I} + 2\mu_e\mathbf{D}, \qquad \mathbf{D} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \tag{10}$$

and where the *dynamic viscosity* μ_0 has been substituted by the *effective viscosity* μ_e depending on the norm of the strain rate tensor $\mathbf{D}$. To fix ideas, let us choose the following effective viscosity

$$\mu_e = \mu_e(|\mathbf{D}|) = \mu_0\left(1 + \lambda^2|\mathbf{D}|^2\right)^q \tag{11}$$

where $q > 0$ and $\lambda > 0$.

Remark 1. If we take $q = 0$ or $\lambda = 0$, Eqs. (8)–(11) are the Navier–Stokes equations. Otherwise, we obtain a generalization of the Navier–Stokes equations, that is frequently used to model non-Newtonian flow. There exists extensive literature on this subject (see, for example, [14–18]). Eqs. (8)–(11) are also known as Ladyzhenskaya model, and a unique global weak solution is guaranteed if $q \geq 1/4$ (see [1]). We obtain the Smagorinsky model if we take $q = 1/2$, and choosing the value of λ properly, we can recover the dissipation predicted by Kolmogorov.

From the above remark, let us consider the effective viscosity given by (11) with $q = 1/2$ in what follows. Thus, we introduce the dissipation predicted by Kolmogorov as part of the model, and we shall approach the subgrid-scale stress tensor in another way (see (21)–(22)).

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