Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Finite difference approximation of space-fractional diffusion problems: The matrix transformation method

Béla J. Szekeres, Ferenc Izsák *

Department of Applied Analysis and Computational Mathematics & MTA ELTE NumNet Research Group, Eötvös Loránd University, Pázmány P. stny. 1C, 1117 - Budapest, Hungary

ARTICLE INFO

Article history:
Received 29 June 2016
Received in revised form 10 October 2016
Accepted 15 November 2016
Available online xxxx

Keywords:
Fractional-order diffusion
Matrix transformation method
Finite difference method

ABSTRACT

A mathematical analysis is presented to establish the convergence of the matrix transformation (or matrix transfer) method for the finite difference approximation of space-fractional diffusion problems. Combined this with an implicit Euler time discretization, the optimal order convergence is proved with respect to the discrete L_2 and the maximum norm. The analysis is performed on general two and three-dimensional domains with homogeneous boundary conditions. The corresponding error estimates are illustrated with some numerical experiments.

1. Introduction

Numerical solution of space-fractional diffusion problems has been studied extensively in the last decade. The finite difference methods for the conventional diffusion problems were extended in some sense including the development of higher-order methods for the spatial discretization [1,2] and the time integration [3], generalization of ADI methods [4,5], construction of appropriate iterative solvers [6] and computing on non-uniform meshes [7]. On the development of the computational efficiency, we refer to [5,7].

Usually, the first step of the numerical solution is the discretization of the fractional diffusion operator. Initiated by the work [8], many authors contributed to this by developing high-order [9] or compact [10] finite difference approximations. Another possibility is the finite element discretization, which using a dimensional lifting is fully analyzed in [11]. The non-trivial aspect of the finite difference approximations is that we have to use a wide stencil for the approximations due to the non-local nature of the corresponding differential operators. This results in full matrices with non-trivial matrix entries. Also, the coefficients of a straightforward approximation have to be shifted to ensure the stability in the time integration [8].

A favorable alternative to bypass this procedure is offered by the so-called matrix transformation method, which was first proposed in [12,13]. According to this, we simply have to take the power of the matrix corresponding to the conventional diffusion (negative Laplacian) operator. For the computation of this matrix [14] or immediately solving the linear systems in the time integration, efficient techniques have been proposed [15].

But can we establish the convergence of this simple approach? A corresponding analysis is only available in case of finite element discretizations [16] and for cubic domains in case of finite difference approximations.

The aim of this contribution is to prove a general convergence result of the matrix transformation (MTM) method for finite difference approximation of space-fractional diffusion problems in two and three space dimensions.

* Corresponding author.
E-mail addresses: szbpagt@cs.elte.hu (B.J. Szekeres), izsakf@cs.elte.hu (F. Izsák).

http://dx.doi.org/10.1016/j.camwa.2016.11.021

After the formal problem statement, we collect some tools which will be used in the error analysis. The estimates for the Laplacian eigenfunctions and eigenvalues are of central importance. We perform then the error analysis verifying the conditions of the Lax equivalence theorem. The article is closed with some numerical experiments illustrating the convergence results.

2. Mathematical preliminaries

The equation to solve. We investigate the finite difference numerical solution of the space-fractional diffusion problem

$$\begin{cases}\partial_t u(t,\mathbf{x}) = -\mu(-\Delta_{\mathcal{D}})^\alpha u(t,\mathbf{x}) & \mathbf{x}\in\Omega,\ t>0\\ u(0,\mathbf{x}) = u_0(\mathbf{x}) & \mathbf{x}\in\Omega\end{cases} \tag{1}$$

where $\Delta_{\mathcal{D}}$ denotes the Laplacian operator on the computational domain $\Omega \subset \mathbb{R}^d$ with homogeneous Dirichlet boundary conditions, which are implicitly prescribed in this way. Using the compact embedding $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$, we have that $\Delta_{\mathcal{D}}^{-1} : L_2(\Omega) \to L_2(\Omega)$ is positive, compact and self-adjoint, such that the power in (1) makes sense. Here, μ is a positive diffusion coefficient, $d = 2, 3$ is the space dimension and $u_0 : \Omega \to \mathbb{R}$ is given. Since we apply a finite difference approach, we assume at this stage that $u_0 \in C_0^2(\Omega)$ to have a well-defined classical Laplacian, which is approximated first.

Note that various operators are available for modeling space-fractional diffusion problems. The favor of using the fractional Laplacian on the right-hand side of (1) is that this operator is arising from discrete stochastic models [17], which correspond to real-life observations. Also this can be recognized as a special non-local operator, which satisfies a modified Fick's law with mass conservation [18]. For alternative definitions of fractional order derivatives, we refer to [19,20] and a detailed comparison of them can be found in [21].

Eigenfunction expansion, the fractional Laplacian and an embedding theorem. The Hilbert–Schmidt theory gives that the eigenfunctions $\{\phi_j\}_{j\in\mathbb{Z}^+}$ of $-\Delta_{\mathcal{D}}$ form a complete orthogonal system in $L_2(\Omega)$ with the associated eigenvalues $0 < \lambda_1 \le \lambda_2 \le \dots$. With these

$$u = \sum_{j=1}^{\infty} u_j\phi_j \tag{2}$$

denotes the Fourier expansion of u provided that $\|\phi_j\|_{L_2(\Omega)} = 1$.

The fractional Laplacian is defined then on the linear space

$$D_{2\alpha} := \left\{ u \in L_2(\Omega) : \sum_{j=1}^{\infty} u_j^2\lambda_j^{2\alpha} < \infty \right\}$$

with

$$(-\Delta_{\mathcal{D}})^\alpha u := \sum_{j=1}^{\infty} u_j\lambda_j^\alpha\phi_j \quad \text{and} \quad |u|^2_{D_{2\alpha}} := \sum_{j=1}^{\infty} u_j^2\lambda_j^{2\alpha}. \tag{3}$$

We will make use of a classical Sobolev embedding theorem $H^6(\Omega) \subset C^4(\Omega)$, which implies that

$$\|u\|_{C^4(\Omega)} \lesssim \|u\|_{H^6(\Omega)}. \tag{4}$$

For the general statement and the proof, we refer to [22], Theorem 4.12.

The notation $A \lesssim B$ means that there is a mesh-independent constant c such that $A \le cB$ for the (usually mesh-dependent) quantities A and B. If both $A \lesssim B$ and $B \lesssim A$ are satisfied then we simply write $A \approx B$.

Estimates for Laplacian eigenvalues and eigenfunctions. The asymptotic behavior of the series $(\lambda_j)_{j\in\mathbb{Z}^+}$ can be given as

$$\lambda_k \approx k^{\frac{2}{d}}, \tag{5}$$

see [23]. For the maximum of $|\phi_m|$, we have the following estimate:

$$\max_{\Omega} |\phi_k| \le \lambda_k^{\frac{d}{4}}, \tag{6}$$

see [24]. For an exhaustive review of similar results, we refer to [25].

We also need a statement on the regularity of the eigenfunctions $\{\phi_j\}_{j\in\mathbb{N}^+}$. We recall a simplified version of Theorem 3.1 in [26].

If Ω is a bounded Lipschitz domain then there is an index j_0 such that for all $j \ge j_0$ and $n \in \mathbb{N}$, we have

$$\|\nabla^{n+2}\phi_j\|_0 \lesssim \lambda_j^{\frac{n}{2}+1}. \tag{7}$$

Please cite this article in press as: B.J. Szekeres, F. Izsák, Finite difference approximation of space-fractional diffusion problems: The matrix transformation method, Computers and Mathematics with Applications (2016), http://dx.doi.org/10.1016/j.camwa.2016.11.021