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Local and parallel finite element post-processing scheme for the Stokes problem

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ABSTRACT

A local and parallel finite element post-processing scheme based on partition of unity method is proposed and analyzed in this paper for the Stokes problem. Firstly, a standard Galerkin finite element method on a relatively coarse grid is used to obtain the approximation of the lower frequency components. Secondly, the higher frequency components are computed on fine grids by some local and parallel procedure to post-process the standard Galerkin approximation. The motivation of the proposed local and parallel finite element post-processing scheme is based on the superposition principle. Finally, to eliminate the effect of the Dirichlet boundary conditions which are imposed on the internal artificial boundaries, a global coarse grid correction is done to improve the L^2-accuracy of the approximation.

1. Introduction

Post-processing and parallel computing are powerful techniques for numerical simulation of solutions to PDEs with high resolution. In previous works [1–4], Hou presented a two grid post-processing Galerkin method in weakly coupled form by taking advantage of some system relevant projections. The decomposition of the solution and therefore the solution space in these papers is based on a new projection defined according to the standard Galerkin approximation in the coarse grid subspace.

In this paper, by using the ideas of [5,6], a local and parallel finite element post-processing scheme is proposed for the Stokes problem. The main philosophy behind this paper is that we could treat different scales by different tools. Such idea was successfully used in the multi-grid, domain decomposition and nonlinear Galerkin methods (see [7–12]). Since the higher frequency components decay very fast and behave more locally, we can approximate such components by a series of locally defined approximate residual problems with homogeneous Dirichlet boundary conditions which therefore can be solved in parallel. Such local and parallel algorithm, to our knowledge, was first proposed in [6] for linear elliptic boundary value problems and extended to nonlinear elliptic boundary value problems in [13]. Then these local and parallel finite element algorithms were extended for the Stokes and Navier–Stokes equations by He et al. [14–16] and Zheng et al. [17–20], and other equations [21–24]. In particular, Yu et al. [17] proposed one parallel partition finite element for the Stokes problem, this method is developed by combining the local and parallel algorithm with the partition of unity method, that is, they firstly solved the Stokes problem with the local and parallel algorithm, then used the partition of unity method to collect all the local solutions to obtain one globally continuous solution. The key issue for constructing local and parallel scheme is to

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localize the post-processing procedure. The technical tool for localization in the previously mentioned references is a local error estimate for finite element approximations.

In this paper, we follow the basic idea presented in [6] to construct the local and parallel finite element post-processing scheme. The main difference is the motivation of the proposed method is based on the superposition principle of linear systems. In multi-grid method, whether linear or nonlinear problems, after getting the global lower frequencies, the higher frequencies are usually approximated by certain linear residual equations on finer grid. By superposition principle, the global higher frequency residual equation can be equivalent to the sum of a series of simple independent sub-problems defined globally, for example, each sub-problem is driven only by free term with a very small support. These sub-problems can be regarded as certain local "residual" equations. Since the solutions to such sub-problems of "residual", which are driven by free terms with very small compact supports, approach zero rapidly away from the small compact supports (for numerical illustration, we refer readers to [5]), we restrict such sub-problems onto some relatively small regions with the small compact supports of the free terms in them to achieve the localization. Such localization technique is widely used for elliptic boundary value problems, for example see [5,25] and etc. Besides the local parallel property, the merit of the proposed method based on superposition principle is that the global post-processing solution is the direct summation of the lower frequency and every solution to the approximate local "residual" equations. Noticing that each sub-problem is a linear equation with homogeneous Dirichlet boundary condition, the post-processed solution will keep continuous.

The rest of the paper is organized as follows. In the coming section, some preliminary materials are provided. In Section 3, local and parallel finite element post-processing scheme is constructed. Error estimates in both H^1 and L^2 norms are obtained for the proposed method in Section 4. Finally, some numerical experiments are given to support our analysis in Section 5.

2. Preliminaries

For a bounded domain, we use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms, see, e.g., [26,27]. For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, $\|\cdot\|_{s,\Omega} = \|\cdot\|_{s,2,\Omega}$. In some places of this paper, $\|\cdot\|_{s,\Omega}$ should be viewed as piecewise defined if it is necessary. For convenience, following [12,6], the symbols $\lesssim$, $\gtrsim$ and $\approxeq$ will be used in this paper. That $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$ and $x_3 \approxeq y_3$, mean that $x_1 \leq C_1 y_1$, $x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$ for some constants C_1, c_2, c_3 and C_3 that are independent of mesh size. In the following, we denote by $(\cdot,\cdot)$ the L^2-inner product on Ω. Thus, $\|\cdot\|_{0,\Omega} = (\cdot,\cdot)^{\frac{1}{2}}$ and, in $H_0^1(\Omega)$, we know that $\|\cdot\|_{1,\Omega} \approxeq \|\nabla\cdot\|_{0,\Omega}$. For subdomains $S_1 \subset S_2 \subset \Omega$, we write $S_1 \subset\subset S_2$ to mean that $\mathrm{dist}(\partial S_2 \setminus \partial\Omega, \partial S_1 \setminus \partial\Omega) > 0$. The space $H^{-1}(\Omega)^d$, the dual of $H_0^1(\Omega)^d (d = 2, 3)$,will also be used. In the rest of this paper, we always use $d = 2, 3$ to denote the space dimension of the domain Ω.

2.1. The Stokes equations

We consider the following Stokes equations defined on a smooth domain $\Omega \subset R^d$, $d = 2, 3$:

$$\begin{cases} -\triangle u + \nabla p &= f \quad \text{in } \Omega, \\ \mathrm{div}\, u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $u = (u_1, \ldots, u_d)$ is the velocity, p is the pressure, f represents the density of body forces.

In order to introduce a variational formulation, we set

$$a(u, v) = (\nabla u, \nabla v) \quad \forall u, v \in H^1(\Omega)^d.$$

For a given $f \in H^{-1}(\Omega)^d$, the weak form of (2.1) reads: find a pair of $[u, p] \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$B([u, p]; [v, q]) = (f, v) \quad \forall v \in H_0^1(\Omega)^d, \ \forall q \in L_0^2(\Omega), \tag{2.2}$$

where

$$B([u, p]; [v, q]) = a(u, v) - (\mathrm{div} v, p) + (\mathrm{div} u, q). \tag{2.3}$$

Since

$$a(u, v) = (\nabla u, \nabla v) \lesssim \|u\|_1 \|v\|_1,$$
$$(\mathrm{div} v, p) \lesssim \|v\|_1 \|p\|_0, \qquad (\mathrm{div} u, q) \lesssim \|u\|_1 \|q\|_0,$$

we can obtain the bound of $B([u, p]; [v, q])$ with little problem

$$|B([u, p]; [v, q])| \lesssim (\|u\|_1 + \|p\|_0)(\|v\|_1 + \|q\|_0). \tag{2.4}$$

It is well known [28–30] that $B([u, p]; [v, q])$ satisfies the following inf–sup condition with a positive constant β

$$\beta(\|u\|_1 + \|p\|_0) \leq \sup_{[v,q] \in S(\Omega)} \frac{B([u, p]; [v, q])}{\|v\|_1 + \|q\|_0}, \tag{2.5}$$

which implies that problem (2.2) has a unique solution.