Computers and Mathematics with Applications 73 (2017) 163–172

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Iterative Tikhonov regularization for the Cauchy problem for the Helmholtz equation

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ARTICLE INFO

Article history:
Received 8 April 2016
Received in revised form 30 August 2016
Accepted 7 November 2016
Available online 1 December 2016

Keywords:
Helmholtz equation
Cauchy problem
Adjoint method
Inverse problem
Ill-posed problem
Tikhonov regularization

ABSTRACT

The Cauchy problem for the Helmholtz equation appears in various applications. The problem is severely ill-posed and regularization is needed to obtain accurate solutions. We start from a formulation of this problem as an operator equation on the boundary of the domain and consider the equation in $(H^{1/2})^*$ spaces. By introducing an artificial boundary in the interior of the domain we obtain an inner product for this Hilbert space in terms of a quadratic form associated with the Helmholtz equation; perturbed by an integral over the artificial boundary. The perturbation guarantees positivity property of the quadratic form. This inner product allows an efficient evaluation of the adjoint operator in terms of solution of a well-posed boundary value problem for the Helmholtz equation with transmission boundary conditions on the artificial boundary.

In an earlier paper we showed how to take advantage of this framework to implement the conjugate gradient method for solving the Cauchy problem. In this work we instead use the Conjugate gradient method for minimizing a Tikhonov functional. The added penalty term regularizes the problem and gives us a regularization parameter that can be used to easily control the stability of the numerical solution with respect to measurement errors in the data. Numerical tests show that the proposed algorithm works well.

1. Introduction

The Helmholtz equation arises in a wide range of applications related to acoustic and electromagnetic waves. Depending on the type of the boundary conditions, it appears in inverse problems for the determination of acoustic cavities [1], the detection of the source of acoustical noise [2,3], the description of underwater waves [4], the determination of the radiation field surrounding a source of radiation [5], and the localization of a tumor in a human body [6].

We consider the inverse problem of reconstructing the acoustic or electromagnetic field from inexact data given only on an open part of the boundary of a given domain. The governing equation for such a problem is the Helmholtz equation. In order to formulate the problem, we let Ω be a bounded domain in $\mathbb{R}^d$ with a Lipschitz boundary Γ divided into two disjoint parts Γ_0 and Γ_1 such that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$ is Lipschitz; see Fig. 1.

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http://dx.doi.org/10.1016/j.camwa.2016.11.004
0898-1221/

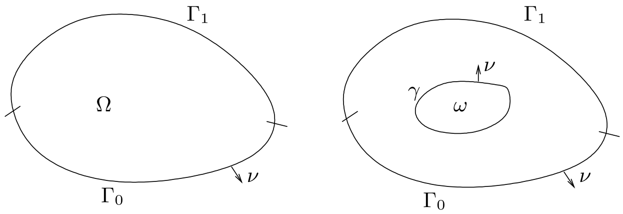

Fig. 1. Description of the domain considered in this paper with a boundary Γ divided into two parts Γ_0 and Γ_1. On the right, an interior boundary γ_i has been introduced.

The Cauchy data ϕ and ζ are only known on Γ_0, possibly with a certain noise level. We can thus formulate the problem as follows:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = \phi & \text{on } \Gamma_0, \\ \partial_\nu u = \zeta & \text{on } \Gamma_0, \end{cases} \tag{1.1}$$

where the wave number k is a positive real number, ν is the outward unit normal of the boundary Γ, and ∂_ν denotes the outward normal derivative. This is the Cauchy problem for the Helmholtz equation and it is severely ill-posed [7–9]. Hence, regularization is needed in order to solve the problem accurately. Different regularization methods have been suggested by various authors. We mention for instance the potential function method [10], the modified Tikhonov regularization method [7,11], the truncation method [12], the method of approximate solutions [5], the wavelet moment method [13], the conjugate gradient methods with the boundary element method [14,15], and the alternating boundary element method [8].

By taking advantage of the linearity of Eq. (1.1) we can, e.g., set $\phi = 0$ and consider the following well-posed problem: Find $u \in H^1(\Omega)$ such that

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ \partial_\nu u = \eta & \text{on } \Gamma_1, \end{cases} \tag{1.2}$$

where we have imposed Neumann conditions on Γ_1 and zero Dirichlet conditions on Γ_0. Using the above well-posed problem we define an operator

$$K\eta = \partial_\nu u|_{\Gamma_0}, \tag{1.3}$$

and we have effectively reformulated the Cauchy problem (1.2) as a linear operator equation $K\eta = \zeta$. The problem with such an approach is that the formulation (1.2) can have non-zero solutions even for $\eta = 0$, or the solution in the interior may have a large norm due to clustering of eigenvalues near k^2.

The operator equation (1.3) can be considered in weighted L^2 spaces, see [16], or using $H^{1/2}$ spaces. We consider the second option and the space for η is $H^{1/2}(\Gamma_1)^*$. Then the solution u belongs to $H^1(\Omega)$ and $K\eta$ lies in $H^{1/2}(\Gamma_0)^*$. Despite the fact that both spaces are Hilbert it is problematic to find an expression for the adjoint K^* in terms suitable for applications. It is important to find inner products for these spaces that allows the efficient evaluation of the adjoint operator.

In the case $k = 0$, i.e. the Laplace equation, we can use an equivalent Hilbert norm in $H^{1/2}(\Gamma_1)^*$,

$$\|\eta\|_{H^{1/2}(\Gamma_1)^*} = \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2}, \tag{1.4}$$

where u is harmonic and $u = 0$ on Γ_0 and $u = \eta$ on Γ_1. The adjoint operator, with respect to the corresponding inner product, can be expressed through the solution of a boundary value problem for the Laplace equation. For the construction of the adjoint it is important to note that (1.4) is the bilinear form corresponding to the differential operator in (1.2) for $k = 0$.

If $k \neq 0$ then the bilinear form corresponding to (1.2) may change sign. In [17,18] we have suggested to use instead,

$$\int_\Omega (|\nabla u|^2 - k^2 u^2)^2 dx + \mu \int_\gamma u^2 ds, \tag{1.5}$$

where γ is a surface in $\bar{\Omega}$ and μ is a positive constant. If γ and μ are chosen so that the form (1.5) is positive for $u \neq 0$ then the adjoint, with respect to the corresponding inner product, could also be defined in terms of a boundary value problem for the Helmholtz operator with suitable transmission conditions on γ.

The alternating iterative algorithm for the Cauchy problem for the Laplace equation, suggested by the authors in [19,20], consists of solving two related Neumann–Dirichlet problems in sequence. The alternating algorithm can be seen as solving the operator Eq. (1.3) by the Landweber method and the two mixed Dirichlet–Neumann problems can be viewed as the application of either the operator K or its adjoint K^*. It has been shown in [17, Section 1.3] that the alternating algorithm