Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Trust-region methods for nonlinear elliptic equations with radial basis functions

Francisco Bernal *

INESC-ID\IST, TU Lisbon, Rua Alves Redol 9, 1000-029 Lisbon, Portugal
Center for Mathematics and its Applications, Department of Mathematics, Instituto Superior Técnico. Av. Rovisco Pais, 1049-001 Lisbon, Portugal

ARTICLE INFO

Article history:
Received 13 December 2015
Received in revised form 9 June 2016
Accepted 16 July 2016
Available online xxxx

Keywords:
Kansa's method
Radial basis function
Nonlinear elliptic PDE
Trust-region method
Monge–Ampère
Plateau's problem

ABSTRACT

We consider the numerical solution of nonlinear elliptic boundary value problems with Kansa's method. We derive analytic formulas for the Jacobian and Hessian of the resulting nonlinear collocation system and exploit them within the framework of the trust-region algorithm. This ansatz is tested on semilinear, quasilinear and fully nonlinear elliptic PDEs (including Plateau's problem, Hele–Shaw flow and the Monge–Ampère equation) with excellent results. The new approach distinctly outperforms previous ones based on linearization or finite-difference Jacobians.

1. Introduction

1.1. RBF interpolation

Given the scalar data $u_1, \ldots, u_N$ on a set (called *pointset*) of distinct points $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^d$ (called *centres*), the RBF interpolant is defined as

$$\tilde{u}(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \phi(\|\mathbf{x} - \mathbf{x}_j\|), \tag{1}$$

where the function $\phi(r) : [0, \infty) \to \mathbb{R}$ is the chosen radial basis function (RBF). A few popular RBFs are shown in Table 1. Throughout this paper, $\|\cdot\|$ is always the 2-norm. The coefficients $\alpha_1, \ldots, \alpha_N$ are determined by collocation

$$\begin{bmatrix} \phi(\|\mathbf{x}_1 - \mathbf{x}_1\|) & \cdots & \phi(\|\mathbf{x}_1 - \mathbf{x}_N\|) \\ \vdots & \ddots & \vdots \\ \phi(\|\mathbf{x}_N - \mathbf{x}_1\|) & \cdots & \phi(\|\mathbf{x}_N - \mathbf{x}_N\|) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}, \tag{2}$$

or, more compactly, $[\phi]\vec{\alpha} = \vec{u}$. Thanks to the radial argument of ϕ, $[\phi]$-called the *RBF interpolation matrix*-is symmetric. Guaranteed non-singularity of $[\phi]$ depends on the RBF ϕ being strictly conditionally positive definite (SCPD)-i.e. bound

* Correspondence to: Center for Mathematics and its Applications, Department of Mathematics, Instituto Superior Técnico. Av. Rovisco Pais, 1049-001 Lisbon, Portugal.
E-mail address: francisco.bernal@ist.utl.pt.

http://dx.doi.org/10.1016/j.camwa.2016.07.014
0898-1221/

Please cite this article in press as: F. Bernal, Trust-region methods for nonlinear elliptic equations with radial basis functions, Computers and Mathematics with Applications (2016), http://dx.doi.org/10.1016/j.camwa.2016.07.014

Table 1
d is the space dimension ($\mathbf{x} \in \mathbb{R}^d$). In MATERN (α, c), $K_\nu(t)$ is the modified Bessel function of the second kind. In WC4 (L), $[f(r)]_+ = 0$ if $r \geq L$, $s = 3 + \lfloor d/2 \rfloor$, and $P(t, s) = (s^2 + 4s + 3)t^2 + (3s + 6)t + 3$ (WC4 works up to $d = 3$).

RBFs used in this paper				
RBF	$\phi(r)$	Notation	Support	Convergence rate
Multiquadric	$\sqrt{r^2 + c^2}$	MQ(c)	$r \leq \infty$	Spectral
Inverse multiquadric	$1/\sqrt{r^2 + c^2}$	IMQ(c)	$r \leq \infty$	Spectral
Matérn	$(r/c)^{(\alpha-d)/2} K_{(\alpha-d)/2}(r/c)$	MATERN(α, c)	$r \leq \infty$	Spectral
Wendland C^4	$[1 - (r/L)]_+^{s+2} P(r/L, s)$	WC4(L)	$r \leq L$	Algebraic

to yield positive-definite $[\phi]$. For instance, in Table 1 all the RBFs are SCPD except for the multiquadric, where the RBF interpolant needs to be augmented with a constant to yield a positive definite $[\phi]$ [1,2].

1.2. Kansa's method

In 1990, Kansa adapted this approach to the solution of linear boundary value problems (BVPs) [3,4]. Consider the elliptic BVP

$$\begin{cases} \mathcal{L}^{PDE} u(\mathbf{x}) = f, & \text{if } \mathbf{x} \in \Omega \\ \mathcal{L}^{BC} u(\mathbf{x}) = g, & \text{if } \mathbf{x} \in \partial\Omega, \end{cases} \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 1$, $u : \Omega \to \mathbb{R}$ is smooth, and $\mathcal{L}^{PDE}$ and $\mathcal{L}^{BC}$ are the interior and boundary linear operators, respectively. Kansa's idea was to discretize $\Omega \cup \partial\Omega$ into a pointset $\Xi_N = \{\mathbf{x}_i\}_{i=1}^N$, and look for an approximation $\tilde{u}$ to u with an RBF interpolant like (1). Without loss of generality, we can assume that the first M nodes in Ξ_N belong to the interior of Ω and the last $N - M$ are discretizing its boundary. By linearity, collocation of (3) on that interpolant leads to

$$[\mathcal{L}\phi]\vec{\alpha} := \begin{bmatrix} [\mathcal{L}^{PDE}\phi]_\Omega \\ [\mathcal{L}^{BC}\phi]_{\partial\Omega} \end{bmatrix} \vec{\alpha} := \begin{bmatrix} \mathcal{L}^{PDE}\phi_{11} & \cdots & \mathcal{L}^{PDE}\phi_{1N} \\ \vdots & \ddots & \vdots \\ \mathcal{L}^{PDE}\phi_{M1} & \cdots & \mathcal{L}^{PDE}\phi_{MN} \\ \mathcal{L}^{BC}\phi_{M+1,1} & \cdots & \mathcal{L}^{BC}\phi_{M+1,N} \\ \vdots & \ddots & \vdots \\ \mathcal{L}^{BC}\phi_{N1} & \cdots & \mathcal{L}^{BC}\phi_{NN} \end{bmatrix} \vec{\alpha} = \begin{bmatrix} f(\mathbf{x}_1) \\ \vdots \\ f(\mathbf{x}_M) \\ g(\mathbf{x}_{M+1}) \\ \vdots \\ g(\mathbf{x}_N) \end{bmatrix}. \tag{4}$$

(Check Section 1.7 for the notation.) This method for solving PDEs has many appealing features: it is meshless,very easy to code, appropriate for high-dimensional PDEs (thanks to the radial argument of the RBFs, which is dimension-blind) and-as long as the solution is smooth-enjoys exponential convergence with respect to the fill distance of the pointset Ξ_N (for many RBFs at least, see Table 1). For a complete exposition, the reader is referred to [1]. Regarding solvability, conditions which guarantee that the *differentiation matrix* in (4) be nonsingular have not yet been established. (In fact, there are crafted examples which yield a singular matrix [5]), but such cases should be exceedingly rare, as also confirmed by years of praxis. On the other hand, Kansa's method may lead to very ill-conditioned matrices, meaning that only pointsets with up to a few thousands of nodes can be used before the matrix in (4) becomes numerically singular. Larger problems can be tackled by using compactly supported RBFs such as WC4 in Table 1 (at the expense of sacrificing spectral convergence), by the RBF-QR method [6] (for some RBFs), and/or by using the novel RBF-partition of unity method [7].

1.3. Nonlinear equations

Extending Kansa's method to nonlinear equations is straightforward. Let us introduce the following compact notation for a nonlinear elliptic BVP:

$$\mathcal{W}[\mathbf{x}, u(\mathbf{x}), Du(\mathbf{x})] = 0 \Rightarrow \begin{cases} \mathcal{W}^{PDE} = 0, & \text{if } x \in \Omega \\ \mathcal{W}^{BC} = 0, & \text{if } x \in \partial\Omega, \end{cases} \tag{5}$$

where $Du(\mathbf{x})$ is shorthand notation for any kind of derivatives present in (5), such as $\partial/\partial x$, ∇^2, etc. Collocation of (1) on (5) leads to the nonlinear system

$$W_i(\vec{\alpha}) := \mathcal{W}[\mathbf{x}_i, \tilde{u}(\mathbf{x}_i), D\tilde{u}(\mathbf{x}_i)] = 0, \quad 1 \leq i \leq N. \tag{6}$$

A root $\vec{\alpha}_*$ of (6)-i.e. $\{W_i(\vec{\alpha}_*) = 0\}_{i=1}^N$ or simply $\vec{W} = 0$-represents an RBF solution $\tilde{u}(\vec{\alpha}_*)$ of the BVP (5). Even if the nonlinear BVP (5) has one unique solution, the meshless discretization (6) may have none, one, multiple or infinitely many roots, regardless of the fact that the system is square. Therefore, it is not evident that collocation is the best approach to RBF representations of solutions to nonlinear BVPs, especially given that least-squares RBF approximations have been found