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A new spectral Galerkin method for solving the two dimensional hyperbolic telegraph equation

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ABSTRACT

Telegraph equation is more suitable than ordinary diffusion equation in modeling reaction–diffusion for several branches of sciences and engineering. In this paper, a new numerical technique is proposed for solving the second order two dimensional hyperbolic telegraph equation subject to initial and Dirichlet boundary conditions. Firstly, a time discrete scheme based on the finite difference method is obtained. Unconditional stability and convergence of this semi-discrete scheme are established. Secondly, a fully discrete scheme is obtained by the Sinc-Galerkin method and the problem is converted into a Sylvester matrix equation. Especially, when a symmetric Sinc-Galerkin method is used, the resulting matrix equation is a discrete ADI model problem. Then, the alternating-direction Sinc-Galerkin (ADSG) method is applied for solving this matrix equation. Also, the exponential convergence rate of Sinc-Galerkin method is proved. Finally, some examples are given to illustrate the accuracy and efficiency of proposed method for solving such types of differential equations compared to some other well-known methods.

1. Introduction

In recent years, the hyperbolic partial differential equations have received considerable attention due to their applications in modeling many physical phenomena such as vibrations of structures (e.g., building, beams and machines) and atomic physics [1]. The hyperbolic telegraph equation is among these types of equations. It arises in study of electric signal in a transmission line, dispersive wave propagation, parallel flows of viscous Maxwell fluids [2], pulsating blood flow in arteries and in one dimensional random motion of bugs along a hedge [3]. In this paper, we consider the second order two-space dimensional hyperbolic telegraph equation in the following form [4]

$$u_{tt}(x,y,t)+2\alpha u_t(x,y,t)+\beta^2u(x,y,t)=\Delta u(x,y,t)+f(x,y,t),\quad (x,y,t)\in R\times[0,T], \tag{1}$$

with initial conditions

$$\begin{cases}u(x,y,0)=u_0(x,y),\\ u_t(x,y,0)=v_0(x,y),\end{cases}\quad (x,y)\in R, \tag{2}$$

and Dirichlet boundary conditions

$$\begin{cases}u(a,y,t)=f_1(y,t),\ u(b,y,t)=f_2(y,t),\\ u(x,c,t)=f_3(x,t),\ u(x,d,t)=f_4(x,t),\end{cases}\quad (x,y)\in\partial R\times[0,T], \tag{3}$$

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where R is the rectangle $\{(x, y)|a < x < b,\ c < y < d\}$, Δ is the operator of Laplacian, α and β are constants in which for $\alpha > 0$, $\beta = 0$, Eq. (1) represents a damped wave equation and for $\alpha > 0$, $\beta > 0$, it is called the telegraph equation.

There are several numerical methods in the literature, which have been developed for solving one, two and three dimensional hyperbolic telegraph equations. In [5], a numerical method based on the boundary integral equation (BIE) and an application of the dual reciprocity method (DRM) was presented to solve the second-order one space-dimensional hyperbolic telegraph equation. In [6], a differential quadrature method (DQM) was proposed for the numerical solution of one and two space dimensional hyperbolic telegraph equation subject to appropriate initial and boundary conditions. Jiwari et al. [7] proposed a numerical hybrid method based on the polynomial differential quadrature method (PDQM) and the RK4 method to find numerical solutions of two dimensional hyperbolic telegraph equation with the Dirichlet and Neumann boundary condition. Two classes of meshfree methods based on the radial basis functions, direct and indirect approaches and their localized forms were applied for solving Eq. (1) [8]. Bülbül and Sezer [9], considered a Taylor matrix method for numerical solution of two-space-dimensional linear hyperbolic equation. This method transformed the equation into a matrix equation and, they showed that their method has polynomial convergence rate. Ding and Zhang [10] proposed a three level compact difference scheme of $O(\tau^4 + h^4)$ for the difference solution of Eq. (1). Mohanty and Jain [11] introduced a new unconditionally stable alternating direction implicit (ADI) scheme of second order accurate for two-dimensional telegraph equation. A meshless local weak–strong method based on the local Petrov–Galerkin (MLPG) process [12], a collocation method by employing thin plate splines radial basis functions [13] and the high order implicit collocation method [14] were applied to approximate the solution of Eq. (1). Two and three dimensional second order hyperbolic telegraph equations subject to appropriate initial conditions were studied through an analytical solution procedure using a reliable semi-analytic method, which is called the reduced differential transform method (RDTM) [15]. Two new iterative numerical schemes based on the centered and rotated seven-point finite difference discretizations were proposed for the solution of a three dimensional second order hyperbolic telegraph equation, subject to specific initial and Dirichlet boundary conditions [16]. Authors of [4], presented a new stable approach for solving Eq. (1). Firstly, they used the modified cubic B-spline functions based on differential quadrature method for space discretization. Then, they used SSP-RK43 scheme for solving the resulting system of ODEs.

Since 1979, numerical methods based on the sinc function have been extensively used for solving different types of differential equations due to their fantastic properties such as their fast convergence order which is of exponential order, i.e., $\exp(-cN^{\frac{1}{2}})$, and their flexibility in handling problems with arbitrary domains [17–19]. In this work, we will present a new hybrid method based on the finite difference scheme and Sinc-Galerkin method to approximate the solution of Eq. (1). Firstly, a semi-discrete scheme is obtained by applying the finite difference method on time derivatives. The stability and convergence of this scheme are discussed by using the mathematical induction. Then, we will use the Sinc-Galerkin method to find the approximate solution of time discrete scheme. The resulting matrix equation is in the form of Sylvester equation and we show that when the symmetric Sinc-Galerkin method is used, the resulting Sylvester equation is a discrete alternating direction iteration (ADI) model problem. It will be solved via the alternating-direction Sinc-Galerkin (ADSG) method which was first proposed in [20]. Also, the exponential convergence rate of Sinc-Galerkin method for solving Eq. (1) will be proved.

The rest of this paper is organized as follows. In Section 2, a semi-discrete scheme based on the finite difference method is presented. Also, the stability and convergence of this scheme are analyzed. Section 3 is devoted to the sinc function and some of its properties which will be used hereafter. Implementation of the Sinc-Galerkin method for solving the semi-discrete scheme and the exponential convergence rate of this method are also studied in this section. Numerical experiments are displayed in Section 4 which are in line with the theoretical analysis. Finally, concluding remarks are drawn in the last Section.

2. Time discrete scheme

For discretization of time variable, let

$$t_n = n\tau, \quad n = 0, 1, \ldots, N,$$

where $\tau = \frac{T}{N}$ is the step size of time variable.

In this work, we discretize (1) in time direction by means of the following finite difference schemes

$$u_{tt}(x, y, t^n) = \frac{u^{n+1}(x, y) - 2u^n(x, y) + u^{n-1}(x, y)}{\tau^2} + O(\tau^2),$$
$$u_t(x, y, t^n) = \frac{u^{n+1}(x, y) - u^{n-1}(x, y)}{2\tau} + O(\tau^2),$$
$$u(x, y, t^n) = \frac{u^{n+1}(x, y) + u^n(x, y) + u^{n-1}(x, y)}{3} + O(\tau^2),$$
$$\Delta u(x, y, t^n) = \frac{\Delta u^{n+1}(x, y) + \Delta u^n(x, y) + \Delta u^{n-1}(x, y)}{3} + O(\tau^2).$$

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