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A note on “Linear programming models for a stochastic dynamic capacitated lot sizing problem”

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ABSTRACT

Tempelmeier and Hilger (2015) study the stochastic dynamic lot sizing problem with multiple items and limited capacity. They propose a linear optimization formulation for the problem based on a piece-wise linear approximation of the non-linear functions for the expected backorders and the expected inventory on hand. Building on the work of Tempelmeier and Hilger (2015), we correct an erroneous derivation of the linear optimization problem and propose an improved model.

1. Introduction

In this technical note we study the stochastic dynamic capacitated lot sizing problem (SCLSP). Contrary to its deterministic counterpart, demand is assumed to be randomly distributed from a known probability distribution, in this case the Normal distribution. The problem deals with determining a production plan for $\mathcal{K}$ items $(k = 1, 2, \ldots, \mathcal{K})$ over a finite horizon of T periods $(t = 1, 2, \ldots, T)$. All items are produced on a single resource with limited capacity C_t. We assume a forecast is given for each item k over the planning horizon in terms of the expected demand $\mathbb{E}[d_{kt}] = \mu_{d_{kt}}$ and the related standard deviation $\sigma_{d_{kt}}$ per time period. We consider unfulfilled demand to be put on backorder and hence, there are no lost sales.

Tempelmeier and Hilger (2015) assume the “static uncertainty strategy” of Bookbinder and Tan (1988) applies. Based on this assumption, lot sizes as well as the periods that there is production are determined in advance and this production plan is executed regardless of the actual demand realizations.

We show in this technical note that there is an error in the derivation of the stochastic model by Tempelmeier and Hilger (2015) and if used, would lead to incorrect production plans. In the next section we discuss the model as formulated by Tempelmeier and Hilger (2015) and while doing so we point out the error. Then we explain how this error can be corrected and propose an improved model.

2. Analysis

The deterministic counterpart of the capacitated lot sizing problem can be formulated as in Problem 1.

Problem 1 (CLSP).

$$\text{min.} \sum_{t=1}^{T}\sum_{k=1}^{\mathcal{K}}(s_k^c\gamma_{kt} + h_k^c I_{kt}) \tag{1}$$

$$\text{s.t. } I_{kt} = I_{k,t-1} + q_{kt} - d_{kt} \qquad \forall k, t \tag{2}$$

$$\sum_{k\in\mathcal{K}} t_k^p q_{kt} + t_k^s\gamma_{kt} \le C_t \qquad \forall k, t \tag{3}$$

$$q_{kt} \le M\gamma_{kt} \qquad \forall k, t \tag{4}$$

$$\gamma_{kt} \in \{0, 1\} \qquad \forall k, t \tag{5}$$

$$0 \le q_{kt}, I_{kt} \qquad \forall k, t \tag{6}$$

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In this linear optimization problem the objective is to minimize the setup cost s_k^c and the inventory holding cost h_k^c. The decision variables I_{kt}, γ_{kt} and q_{kt} represent, respectively, the inventory on hand, the setup decision and the production quantity. Constraint 2 represents the inventory balance equation with I_{k0} being set to some initial value. Constraint 3 limits setups and production in time period t by the available capacity C_t. Setting up production for an item k takes t_k^s amount of time, and producing one item k takes t_k^p amount of time. Constraint 4 ensures that the setup variable is set to one if product k is produced in period t (M is a sufficiently large number). Constraint 5 states that γ_{kt} is a binary decision variable. Constraint 6 ensures a lower bound on the production quantity and the inventory on hand.

Since demand is uncertain, a service level constraint is introduced to ensure production. This means that for the expected inventory on hand we obtain,

$$\mathbb{E}[I_{kt}] = Q_{kt} - \mathbb{E}[Y_{kt}] + \mathcal{L}^1_{Y_{kt}}(Q_{kt}) \tag{7}$$

with $Y_{kt} = \sum_{\tau=1}^{t} D_{k\tau}$ denoting the cumulated demand from period 1 up to period t. For the expected backlog we obtain,

$$\mathbb{E}[B^l_{kt}] = \mathcal{L}^1_{Y_{kt}}(Q_{kt}) = \mathbb{E}[\max\{0, Y_{kt} - Q_{kt}\}] \tag{8}$$

with $\mathcal{L}^1_{Y_{kt}}(Q_{kt})$ being the first-order loss function of the random variable Y_{kt} and depending on the cumulative production quantity Q_{kt}. Consult Zipkin (2010) or Rossi et al. (2014) for a derivation that leads to an expression in terms of the standard normal probability density function $\phi(\cdot)$ and the standard normal cumulative density function $\Phi(\cdot)$,

$$\mathcal{L}^1_{Y_{kt}}(Q_{kt}) = \sigma_{Y_{kt}}\Big(\phi(z) - z\big(1 - \Phi(z)\big)\Big) \tag{9}$$

with $z = \frac{Q_{kt} - \mathbb{E}[Y_{kt}]}{\sigma_{Y_{kt}}}$. We can now express the expected backorders, $\mathbb{E}[B_{kt}(Q_{kt})]$, in terms of the expected backlog; that is,

$$\mathbb{E}[B_{kt}(Q_{kt})] = \mathcal{L}^1_{Y_{kt}}(Q_{kt}) - \mathcal{L}^1_{Y_{k,t-1}}(Q_{kt}) \tag{10}$$

The expression for the expected backorders can be used to define the following fill-rate constraint,

$$1 - \frac{\sum_{t=1}^{T} \mathbb{E}[B_{kt}(Q_{kt})]}{\sum_{t=1}^{T} \mathbb{E}[d_{kt}]} \geq \beta^\star, \ \forall k \tag{11}$$

with B_{kt} denoting the backorders for product k in time period t and $\beta^\star$ being the target fill-rate.

After introducing the expected values, Tempelmeier and Hilger (2015) derive the approximate stochastic counterpart of Problem 1 by using a piece-wise linear approximation for both functions. The functions are linearized into L line segments on the relevant interval $[u^0_{kt}; u^L_{kt}]$ where subinterval $[u^{l-1}_{kt}; u^l_{kt}]$ relates to line segment l ($1 \leq l \leq L$). The slope associated with line segment l of the expected inventory on hand function for item k at time period t is as follows,

$$\Delta^l_{I_{kt}} = \Big(\Big(u^l_{kt} - \mathbb{E}[Y_{kt}] + \mathcal{L}^1_{Y_{kt}}(u^l_{kt})\Big) - \Big(u^{l-1}_{kt} - \mathbb{E}[Y_{kt}] + \mathcal{L}^1_{Y_{kt}}(u^{l-1}_{kt})\Big)\Big) \frac{1}{u^l_{kt} - u^{l-1}_{kt}} \qquad \forall k, t, l \tag{12}$$

Similarly, the slope associated with line segment l of the expected backorders function for item k at time period t is as follows,

$$\Delta^l_{B_{kt}} = \Big(\mathcal{L}^1_{Y_{kt}}(u^l_{kt}) - \mathcal{L}^1_{Y_{k,t-1}}(u^l_{kt})\Big) - \Big(\mathcal{L}^1_{Y_{kt}}(u^{l-1}_{kt}) - \mathcal{L}^1_{Y_{k,t-1}}(u^{l-1}_{kt})\Big) \frac{1}{u^l_{kt} - u^{l-1}_{kt}} \qquad \forall k, t, l \tag{13}$$

We now introduce a new decision variable w^l_{kt} to denote the part of the cumulative production quantity in time period t for product k and line segment l. Until line segment $l^\star$, all decision variables w^l_{kt} are equal to the size of the related interval $[u^{l-1}_{kt}; u^l_{kt}]$, decision variable $w^{l^\star}_{kt}$ is only filled partially, and the remaining decision variables $w^l_{kt}, l \geq l^\star$, are set to zero. Therefore, the following equations must hold for these new decision variables,

$$w^l_{kt} = u^l_{kt} - u^{l-1}_{kt}, l = 1, 2, \ldots, l^\star - 1 \tag{14}$$

$$w^l_{kt} = Q_{kt} - u^{l-1}_{kt}, l = l^\star \tag{15}$$

$$w^l_{kt} = 0, l = l^\star + 1, l^\star + 2, \ldots, L \tag{16}$$

Tempelmeier and Hilger (2015) argue that these equations are implicitly satisfied in their model because "the inventory function is convex, w^l_{kt} is only positive if $w^{l-1}_{kt} = u^{l-1}_{kt} - u^{l-2}_{kt}$". We found out that these equations are not satisfied implicitly in their model because there is an advantage in setting those w^l_{kt}'s larger than zero where the slopes $\Delta^l_{B_{kt}}$ times w^l_{kt} contributes the most to the reduction of the expected backorders. This has to do with the fact that the backorder function found in Eq. (10) is non-convex for $t \geq 2$, as stated in the next lemma.

Lemma 1. *The expected backorder function* $\mathbb{E}[B_{kt}(x)]$ *is non-convex for* $t \geq 2$.

Proof of Lemma 1. See Appendix. □

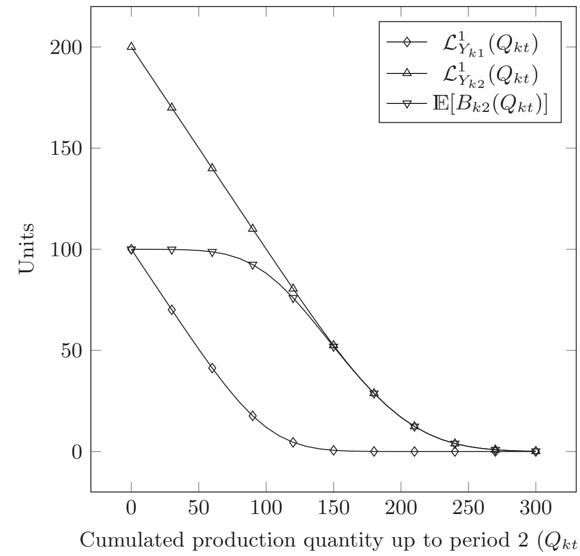

Fig. 1. First-order loss functions and the backorder function for $t = 2$.

Fig. 1 further illustrates this behaviour, it shows a plot of the first-order loss functions $\mathcal{L}^1_{Y_{k,1}}(Q_{kt})$, $\mathcal{L}^1_{Y_{k,2}}(Q_{kt})$ and the expected backorder function $\mathbb{E}[B_{k2}(Q_{kt})]$. From this figure it becomes even clearer that we can significantly reduce the expected backorders while producing less, i.e. fewer w^l_{kt}'s have to be filled to their maximum. With as reason that only those w^l_{kt}'s will be zero that do not add much to a reduction in the expected backorders, while those that contribute the most are filled to their maximum.

We examined the model of Tempelmeier and Hilger (2015) for selected instances of the problem. For example, we solved the problem above for one product, with the mean demand and standard deviation being, respectively, $\mu_{d_{kt}} = 100$ and $\sigma_{d_{kt}} = 30$ for each time period over a horizon of 12 periods. Inventory holding costs are set to $h^c_k = 1$, setup costs to $s^c_k = 500$, setup time is set to zero $t^s_k = 0$, processing time is set to $t^p_k = 1$, capacity is