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Lagrangian and branch-and-cut approaches for upgrading spanning tree problems

Eduardo Álvarez-Miranda[a,*], Markus Sinnl[b]

[a] Department of Industrial Engineering, Universidad de Talca, Curicó, Chile
[b] University of Vienna, Faculty for Business, Economics and Statistics, Department for Statistics and Operations Research, Oskar-Morgenstern-Platz 1, 1090, Vienna, Austria

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Problems aiming at finding budget constrained optimal upgrading schemes to improve network performance have received attention over the last two decades. In their general setting, these problems consist of designing a network and, simultaneously, allocating (limited) upgrading resources in order to enhance the performance of the designed network.

In this paper we address two particular optimal upgrading network design problems; in both cases, the sought-after layout corresponds to a spanning tree of the input network and upgrading decisions are to be taken on nodes. We design Mixed Integer Programming-based algorithmic schemes to solve the considered problems: Lagrangian relaxation approaches and branch-and-cut algorithms. Along with the designed algorithms, different enhancements, including valid inequalities, primal heuristic and variable fixing procedures, are proposed.

Using two set of instances, we experimentally compare the designed algorithms and explore the benefits of the devised enhancements. The results show that the proposed approaches are effective for solving to optimality most of the instances in the testbed, or manage to obtain solutions and bounds giving very small optimality gaps. Besides, the proposed enhancements turn out to be beneficial for improving the performance of the algorithms.

1. Introduction, motivation and problem definition

Different models for finding budget constrained optimal upgrading schemes to improve network performance have been proposed over the last two decades. Roughly speaking, these problems consist of designing a network and, simultaneously, allocating (limited) upgrading resources in order to enhance the network's performance (e.g., cost efficiency, connectivity, stability, etc.). These type of problems are generally referred to as *budget constrained network upgrading problems* (Krumke et al., 1999) and can be found in several decision making contexts. For instance in a multicast communication setting, a backbone server broadcasts a signal to many subscribers; the layout of such communication network should be such that delays between the server and all subscribers must be minimal or bounded. The decision maker seeks for an arrangement of nodes, technologies, and connections so that a positive function of node delays is minimized or it fulfills a Quality-of-Service requirement (Álvarez-Miranda et al., 2016).

In this paper we address network design problems where redundancy is not required, meaning that the sought-after layout has a tree structure to ensure unique connection or communication paths and, in the case of telecommunication or energy transmission applications, to reduce the maintenance and transmission efforts (see Magnanti and Wolsey, 1995; Salama, 1996, for further details). For this setting, we consider two particular optimal upgrading network design problems where upgrading decisions are to be taken on nodes. These problems were originally proposed in Krumke et al. (1999) and motivated by a telecommunication design application and a very-large-scale integration (VLSI) design application. In telecommunication network design, upgrading a node corresponds to installing better communication equipment, which reduces the communication delay for all adjacent edges; while in VLSI design, upgrading a node corresponds to replacing a module at a node, by an equivalent one with better drivers, which decreases the signal transmission delay for all wires connected to the module. The authors provided approximation algorithms and hardness results for the two variants. Another application appears in electric power grids, where transformers and other costly appli-

* Corresponding author.
E-mail addresses: ealvarez@utalca.cl (E. Álvarez-Miranda), markus.sinnl@univie.ac.at (M. Sinnl).

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ances can be located along the network in order improve its performance by reducing the risks of shortages or frequency instability (see Costa et al., 2011, for a related problem).

Before formally defining the addressed problems, we present the considered upgrading model.

Node-based upgrading model. Let $G = (V, E)$ be an undirected graph, where V is the set of nodes and E is the set of edges. For each edge $e \in E$, we are given three integers $d_e^0 \geq d_e^1 \geq d_e^2 \geq 0$; the value d_e^l (the l-upgrade of edge e) represents the *length* or *delay* of e if exactly l of its endpoints are upgraded. This means that upgrading a node $i \in V$ reduces the delay of all the edges that are incident to it. Additionally, for each node $i \in V$, we are given an *upgrade cost* $c_i \geq 0$ which must be paid in case the corresponding node is upgraded. Edge delays as well as node upgrading costs are encoded by vectors $\mathbf{d}$ and $\mathbf{c}$, respectively. As pointed out in Krumke et al. (1999), this model is a generalization of the node upgrade model introduced by Paik and Sahni (1995), in which d_e^1 and d_e^2 are such that $d_e^1 = \alpha d_e^0$ and $d_e^2 = \alpha^2 d_e^0$, for all $e \in E$, for a given $\alpha \in (0, 1)$.

In the following, let MST denote *minimum spanning tree*. We can now present the formal definition of the two problems addressed in this paper.

The minimum delay upgrading MST problem. Let $\mathcal{T}$ be the set of all spanning trees on G. Likewise, let an *upgrading scheme* S to be encoded by a subset $V_S \subseteq V$, so that all nodes in V_S are upgraded, while the remaining ones are not. An upgrading scheme S is feasible if the cost of the upgrading actions induced by S,

$$C(S) = \sum_{i \in V_S} c_i,$$

does not exceed a given *upgrading budget* $B \geq 0$. Let $\mathcal{S}$ denote the family of all upgrading schemes. Let $D(T, S)$ be the total delay of the spanning tree $T \in \mathcal{T}$ under upgrading scheme $S \in \mathcal{S}$. Using this notation, we can formulate the minimum delay upgrading MST problem (MDUMST) as

$$(T^*, S^*) = \arg\min \{D(T, S) \mid C(S) \leq B, \ S \in \mathcal{S} \text{ and } T \in \mathcal{T}\}; \tag{MDUMST}$$

in other words, we look for a pair (T^*, S^*) that minimizes total delay of the corresponding minimum spanning tree. In Krumke et al. (1999) the authors provided an $(1, (1+\epsilon)^2\mathcal{O}(\log|V|))$-approximation algorithm. In a bicriteria optimization context, this means that the algorithm (i) either produces a solution for which the value $C(S)$ is most $(1+\epsilon)^2\mathcal{O}(\log|V|)$ times the specified budget B, and the value of $D(T, S)$ is the minimum value of a solution that satisfies the budget constraint; or (ii) correctly provides the information that there is no subgraph which satisfies the budget constraint $C(S) \leq B$.

The minimum cost upgrading MST problem. Let $W \geq 0$ be a maximum total delay bound; we seek for a spanning tree $T \in \mathcal{T}$ and an upgrading scheme $S \in \mathcal{S}$, so that the total upgrading cost is minimized while the total delay is at most W, i.e.,

$$(T^*, S^*) = \arg\min \{C(S) \mid D(T, S) \leq W, \ S \in \mathcal{S} \text{ and } T \in \mathcal{T}\}. \tag{MCUMST}$$

Similar as for the MDUMST, an $((1+\epsilon)^2\mathcal{O}(\log|V|), 1)$-approximation algorithm was provided in Krumke et al. (1999).

Previous work. One of the very first references to network upgrading problems based on node-upgrades can be found in Paik and Sahni (1995). In that paper, five variants of network upgrading problems are studied. For these variants delays can be caused both along edges and across nodes, so the upgrade decisions involve both components. The computational complexity of the discussed problems is provided, showing that they range from polynomially solvable problems up to NP-hard problems.

In Ibaraki et al. (2005), the authors study a problem in which the goal is to find a budget constrained node-upgrading decision so as to minimize the *eccentricity*, i.e., the largest distance from one designated node to the other nodes of a tree. Hardness results as well as approximation algorithms are provided for two different node-upgrading models.

Complementary, problems where the designed network corresponds to a path have been also considered before. In Dilkina et al. (2011), the upgrading shortest path problem (USP) is introduced. The problem is presented in the context of improving landscape connectivity within natural reserves. In the USP, the shortest-path cost is based on the cost of the nodes in the path, which can be improved by upgrading, i.e., the node-upgrading model is different to the model considered in this work. A Mixed Integer Programming (MIP) model is provided as well as a greedy heuristic. Computational results show that the proposed heuristic is competitive when compared to the direct resolution of the proposed MIP model in a solver. Later on, in Álvarez-Miranda et al. (2016), both a branch-and-cut (B&C) and a Benders decomposition are designed for the USP. Numerical results show that the designed algorithms outperform those proposed in Dilkina et al. (2011) on equivalent benchmark instances.

It is interesting to note that network upgrading problems can be seen as two-player cooperative games; one of the players wants to efficiently allocate resources for upgrading the network (e.g., S) so that the performance of the decision of a second player (e.g., T) is enhanced. This differs from the *network interdiction* problems (see, e.g. Hemmecke et al., 2003; Israeli and Wood, 2002). In these problems, one of the players aims at efficiently allocating resources in order to *downgrade* or *interdict* the network, which leads to worsening the decision of a second player.

Our contribution and paper outline. In this paper, we propose MIP models for both MDUMST and MCUMST and design specially tailored Lagrangian relaxation and B&C approaches to solve the proposed formulations. We also present different enhancements for the devised approaches, including valid inequalities, primal heuristics and variable fixing procedures.

The proposed algorithmic schemes are shown to be effective for solving to optimality most of the instances in the testbed, or finding solutions and bounds yielding, in general, (very) small optimality gaps. Besides, the proposed enhancements turn out to be beneficial for improving the performance of the algorithms. The Lagrangian relaxation approaches allow us to tackle even larger instances, whose sizes represent a burden for the B&C algorithm.

The paper is organized as follows. In Section 2 we present MIP formulations for both problems and Lagrangian approaches for providing both upper and lower bounds. Enhancements for the approaches are also discussed in the Section. Alternative MIP formulations based on a transformation of the input graph and corresponding B&C approaches are presented in Section 3. In Section 4 we report computational results obtained when solving two sets of instances; one comprised by randomly generated instances, and a second one of instances adapted from the well-known SteinLib dataset. Conclusions and paths for future work are discussed in Section 5.

2. Lagrangian relaxation approaches

For a combinatorial optimization problem (COP), which can be formulated as MIP with a set of *easy* constraints $A\mathbf{x} \leq \mathbf{b}$ and *com-*