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Improving polynomial estimation of the Shapley value by stratified random sampling with optimum allocation

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ABSTRACT

In this paper, we propose a refinement of the polynomial method based on sampling theory proposed by Castro et al. (2009) to estimate the Shapley value for cooperative games. In addition to analyzing the variance of the previously proposed estimation method, we employ stratified random sampling with optimum allocation in order to reduce the variance. We examine some desirable statistical features of the stratified approach and provide some computational results by analyzing the gains due to stratification, which are around 30% on average and more than 80% in the best case.

1. Introduction

One of the most important one-point solution concepts in the framework of coalitional games with side payments is the Shapley value (Shapley, 1953), which proposes the allocation of resources in multiperson interactions by considering the power of the players during their various opportunities for cooperation. Since the pioneering work by Shapley, the Shapley value has been studied widely from a theoretical viewpoint. Many studies have also focused on the potential applications of the Shapley value to specific problems (e.g., see Lucchetti et al., 2010; Moretti, 2010). However, since computing the Shapley value is an NP-complete problem (see Deng and Papadimitriou, 1994; Fernández et al., 2002, or Faigle and Kern, 1992, for more details), the problem of its calculation must be addressed before it can employed as a useful tool in real situations.

Two main approaches have been employed to address the calculation of the Shapley value: developing efficient strategies to compute the Shapley value exactly.

Different approaches have been considered for the exact calculation of the Shapley value. In order to compute the Shapley value exactly, some specific features of the problem and/or the game are assumed. For instance, the Shapley value was obtained in a polynomial manner for some voting games by Bilbao et al. (2000). (Granot et al., 2002) developed a polynomial algorithm to obtain the Shapley value of an *extended tree game*, where a tree network serving heterogeneous customers was constructed. (Castro et al., 2008) computed the Shapley value for an airport game in polynomial time by considering that this value was obtained using the serial cost sharing rule. In addition, Deng and Papadimitriou (1994), Bolus (2011), Chalkiadakis et al. (2012), and Conitzer and Sandholm (2006) developed efficient strategies for computing the Shapley value by finding alternative efficient *representations* for transferable utility (TU) games. However, although these methods aim to serve as general tools, their applications are constrained to some specific classes of games, particularly weighted majority games and combinatorial optimization games. Moreover, all of these methods require a specific representation of the game.

By contrast, few studies have focused on the approximation of the Shapley value. Considering the widespread application of game theory to real-world problems where exact solutions are often not possible, then it is necessary to develop algorithms that may facilitate this approximation.

(i) To the best of our knowledge (also see Maleki et al., 2014), the first attempt to estimate the Shapley value for a large class of games was the algorithm proposed by

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Castro et al. (2009), who suggested the estimation of the Shapley value (or any semivalue) for cooperative games based on sampling. The only constraint on this algorithm being polynomial is the fact that the worth of any coalition must be computed polynomially. This approach generalizes the earliest approximation method proposed by Mann and Shapley (1960) for estimating the *power index* of a weighted voting game. In fact, Mann and Shapley (1960) stated that: "*These methods have potential applications beyond the particular problem solved here*." Moreover, as proposed in the present paper, Mann and Shapley also considered alternative sampling methods for reducing the variance of the basic Monte Carlo sampling approach.

(ii) If an efficient *general purpose* algorithm is available, then it will be possible to estimate the Shapley value by refining the original procedure by Castro et al. in order to exploit the specific properties of the problem considered. Thus, Maleki et al. (2014) proposed the use of stratified random sampling to reduce the estimated error in the original method described by Castro et al. They also considered the same strata treated in the present paper. However, their allocation method ignores the specific features of the game, thereby resulting in an estimation method that improves the original method based on simple random sampling, but it general performs poorly compared with the refinement that we propose in the present study. The reader may refer to Section 6 for a comparative analysis using the cooperative games analyzed by Castro et al. (2009).

Few studies have developed approximations to estimate the Shapley value and most of them are problem specific. Fatima et al. (2006) presented a randomized polynomial method for determining the approximate Shapley value for weighted voting games and k-majority voting games. Crama and Leruth (2007) also proposed the estimation of the Shapley value for a complex voting game that reflects the structure of shareholdings in complex interlocked shareholding structures based on a Monte Carlo simulation.

In this paper, we propose a refinement based on stratified random sampling, which we use instead of simple random sampling to estimate the Shapley value. As found with the original algorithm, these refined estimations are polynomial if each marginal contribution can be calculated in polynomial time. The remainder of this paper is organized as follows. In Section 2, we provide some basic concepts and notation. We also describe the *ApproShapley* algorithm proposed by Castro et al. (2009), which is the basis of our propose method. In Section 3, we analyze the sources of the variance in the estimations derived from *ApproShapley* and we propose three methods for reducing this variance. First, we propose a general method based on stratified random sampling *St-ApproShapley*, Sh^{st}, and we then determine the optimum allocation of the sample sizes in the corresponding strata *St-ApproShapley-opt*, $Sh^{st,\,opt}$. We also propose a two-stage estimation method that preserves the efficiency of the Shapley value, *St-ApproShapley-eff*, $Sh^{st,\,eff}$, in the case where efficiency is a demanding property. In Section 4, we analyze the estimated error for stratified random sampling with the optimal allocation approach, and we compare it with the estimation method proposed by Maleki et al., which is described in Section 3. In Section 5, we present the proposed estimation algorithm. Finally, in Section 6, we present some computational results to show the improvements obtained using the proposed method for the cooperative games analyzed by Castro et al. (2009). We give our conclusions in Section 7.

2. Preliminaries

In this section, we introduce some sampling notation and theoretical games in order to understand the rest of this paper. We also recall the original sampling algorithm proposed by Castro et al. (2009) upon which our refinement is based.

A cooperative game in coalitional form with side payments, or with TU, is an ordered pair (N, v), where N is a finite set of players and $v: 2^N \to \mathbb{R}$, with $2^N = \{S \,|\, S \subseteq N\}$, is a *characteristic function* on N that satisfies $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S) \in \mathbb{R}$ is the *worth* of coalition S and it represents the reward that coalition S can obtain by itself if all its members act together. We restrict to the case of TU games in the sequel, so we refer to them simply as *games*. For brevity, throughout this paper, the cardinalities of the sets (coalitions) N and S are denoted by appropriate small letters n and s, respectively. In addition, for convenience, we write the singleton $\{i\}$ as i if there is no ambiguity. When the worth $v(S)$ represents the cost that players must be charged, the game is known as a cost-game and we denote these types of games as (N, c).

Let $\mathcal{G}_N$ be the vector space of all the games with a fixed player set N, and identify $(N, v) \in \mathcal{G}_N$ by its characteristic function v if there is no ambiguity. A *value* φ is an allocation that associates with each game $(N, v) \in \mathcal{G}_N$ a vector $\varphi(N, v) \in \mathbb{R}^N$, where $\varphi_i(N, v) \in \mathbb{R}$ represents the *value* of player i, $i \in N$. Shapley (1953) defines his value as follows:

$$Sh_i(v) = \sum_{\substack{S \subseteq N \\ i \notin S}} \frac{(n-s-1)!\; s!}{n!} (v(S \cup \{i\}) - v(S)), \quad i \in N. \tag{1}$$

The value $Sh_i(v)$ of each player, which is a weighted average of his marginal contributions, allows different interpretations, such as the *payoff* that player i receives when the Shapley value is used to predict the allocation of resources in multiperson interactions, or his *power* when averages are used to aggregate the power of players in their various opportunities for cooperation.

Weber (1988) gave an alternative characterization of the Shapley value in terms of all the possible orders of the players, which was employed for estimation by Castro et al. (2009) . Let O: $N \to N$ be a permutation that assigns the player $O(k)$ to each position k and let $\pi(N)$ denote the set of all possible permutations of the player set N. Given a permutation $O \in \pi(N)$, let us denote $Pre^i(O)$ as the set of predecessors of the player i in the order O (i.e., $Pre^i(O) = \{O(1), \ldots, O(k-1)\}$, if $i = O(k)$). In this setting, the *vector of marginal contributions* for a given order $O \in \pi(N)$, $x(O) = (x(O)_i)_{i \in N}$ is defined as:

$$x(O)_i = v(Pre^i(O) \cup i) - v(Pre^i(O)), \quad i \in N.$$

Weber (1988) showed that the Shapley value can be expressed as the following expectation, where it is assumed that all different orders have equal probability:

$$Sh_i(v) = \sum_{O \in \pi(N)} \frac{1}{n!} x(O)_i, \quad i \in N. \tag{2}$$

Thus, since the Shapley value is an expectation, Castro et al. (2009) proposed its statistical estimation. Next, we describe the algorithm that they proposed (emphApproShapley), which is based on a unique simple random sampling process for estimating the Shapley value Sh_i of all players $i \in N$.

1. The population of the sampling process P is the set of all possible orders of players, i.e., $P = \pi(N)$. Each sampling unit represents an order $O \in \pi(N)$.
2. The vector parameter under paper is $Sh = (Sh_i)_{i \in N}$.
3. The characteristic observed in each sampling unit, $O \in \pi(N)$, is the vector of the marginal contributions for that order O, i.e., $x(O) = (x(O)_i)_{i \in N}$.
4. The estimate of the parameter Sh, $\hat{Sh}$, is the mean of the marginal contributions over the sample M, i.e., $\hat{Sh} = (\hat{Sh}_i)_{i \in N}$, where

$$\hat{Sh}_i = \frac{1}{m} \sum_{O \in M} x(O)_i, \quad i \in N.$$