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Discrete Optimization

On the approximation ratio of the Random Chinese Postman Tour for network search

Thomas Lidbetter

Department of Management Science and Information Systems, Rutgers Business School, 1 Washington Park, Newark, NJ 07102, USA

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ABSTRACT

We consider a classic search problem first proposed by Gal in which a *Searcher* randomizes between unit speed paths on a network, aiming to find a hidden point in minimal expected time in the worst case. This can be viewed as a zero-sum game between the Searcher and a time maximizing *Hider*. It is a natural model for many search problems such as search and rescue operations; the search for an enemy, a bomb or weapons in a military context; or predator–prey search. A *Chinese Postman Tour* (*CPT*) is a minimal time tour of the network that searches all the arcs and a *Random Chinese Postman Tour* (*RCPT*) is an equiprobable choice of any given CPT and its reverse. The full class of networks for which a RCPT is optimal is known, but otherwise little is known about the solution of the game except in some special cases that have complicated optimal strategies that would be impractical to implement. The question of how well a RCPT or any other search strategy performs for general networks has never been analyzed. We show that a RCPT has an approximation ratio of 4/3: that is, the maximum expected time it takes to find a point on the network is no greater than 4/3 times that of the optimal search strategy. We then examine the performance of a RCPT in a related search game recently proposed by Alpern in which the Searcher must return to his starting point after finding the Hider.

1. Introduction

Gal (1979) proposed a search problem on a connected network consisting of nodes joined by arcs. A *Searcher*, beginning at a distinguished *root* node of the network moves around with unit speed with the aim of locating a target, or *Hider*, situated at an unknown point of the network. The problem for the Searcher is to find a randomized search plan, or *mixed strategy* that minimizes the expected time to locate the Hider in the worst case. This can be viewed as a zero-sum game between the time minimizing Searcher and the time maximizing Hider. The idea of this game was originally conceived by Isaacs (1965).

This is a natural model of many search problems in a military context, for example the problem of locating a terrorist, a bomb or a weapon cache hidden in a known environment. It is also relevant to search-and-rescue problems and to models of predator–prey interaction in the natural world. In security scenarios, randomization can be exploited to ensure that your actions are not predictable to your opponent. An example of this is the use of randomized security patrolling used in Los Angeles Airport (see Pita et al., 2008). But ease of implementation is also an important factor to consider, especially in a military context. In this paper we strive to balance the desire for randomness and unpredictability with the desire for easily implemented search strategies.

In his original work, Gal (1979) gave a full solution of his game for tree networks (those with no cycles) and Eulerian networks (those containing an Eulerian tour, that is one that traverses each arc of the network exactly once). Gal found that for both these classes of networks it is optimal for the Searcher to identify a minimal time tour, called a *Chinese Postman Tour* (or *CPT*) of the network, and choose with equal probability the CPT or its reverse. This mixed strategy is known as a *Random Chinese Postman Tour* (or *RCPT*), and it is easy to see that the expected time taken for the Searcher using a RCPT to locate the Hider, wherever he is located on the network, will be no more than half the length L of the CPT. This is because if a given CPT finds a point of a network at time t, the reverse of the CPT finds it at time no greater than $L-t$, so that the expected discovery time is at most $(1/2)t+(1/2)(L-t)=L/2$.

Since its original formulation the game described above has received considerable attention, for example in the work of Reijnierse and Potters (1993), Pavlovic (1995) and Gal (2001). In the latter work, Gal finally completely classified those networks for which it was optimal for the Searcher to employ a RCPT. The networks for which this is the case are known as *weakly Eulerian*, and are, roughly speaking, those consisting of a number of disjoint Eulerian cycles which, when each contracted to a single point leave a tree.

E-mail addresses: tlidbetter@business.rutgers.edu, T.R.Lidbetter@lse.ac.uk

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On networks that are not weakly Eulerian, little is known about the optimal search strategy in general. Even very simple networks can have very complicated optimal search strategies. For example, the solution of the game played on the “3-arc network”, consisting of 2 nodes joined by 3 unit length arcs was discovered by Pavlovic (1993) not until several years after the model was first formulated. For this network, the optimal strategy for the Searcher finds the Hider in expected time $(4+\ln(2))/3 \approx 1.564$, and the optimal Searcher strategy is complicated, involving randomizing between a continuum of strategies. This would be impractical to implement, and so we may compare the performance of the optimal strategy of the Searcher to that of a RCPT, which is straightforward to implement. In order to do this we calculate the *approximation ratio* of a RCPT: that is, the ratio between the maximum expected time required by the RCPT to find a point on the network to that of an optimal strategy. Since the length of a CPT for the 3-arc network is 4, a RCPT finds the Hider in expected time no more than $(1/2)\cdot 4 = 2$, so the RCPT has approximation ratio $2/1.564 \approx 1.28$ for this network.

In this paper we analyze the effectiveness of the RCPT as a search strategy in the general case, giving a simple formula in Section 3 for calculating the approximation ratio of a RCPT and showing that it never exceeds 4/3. It is well known (Edmonds & Johnson, 1973) that the problem of finding a CPT is computable in polynomial time (cubic in the number of nodes of the network), and once it has been found, a RCPT tour is easily implemented in a practical search situation by means of a toss of a coin. This work shows that the RCPT is not only easily implementable but also reasonably efficient as a search strategy. Little attention has been paid in the literature to the problem of finding approximately optimal Searcher strategies in unsolved cases of this game, and for search games in general.

In Section 4, we go on to consider a natural variant on the original model in which the Searcher wishes to minimize not simply the time to find the Hider but the total time to find him and return to her starting point. The return speed may be different to the search speed. This model, known as *find-and-fetch search*, was introduced recently by Alpern (2011a), and is a more appropriate model for search-and-rescue operations in which a casualty must be found and taken back to the hospital in least possible time. The model also pertains to foragers in the natural world who seek food to return to their lairs. This added detail complicates the solution to the model considerably, and in Alpern (2011a) the solution is given for only some classes of tree networks. Unlike in Gal's original model, the Searcher's optimal strategy in the find-and-fetch model is complicated and involves making a randomized decision at each node of the tree. In this work we assess the performance of a RCPT in the find-and-fetch model for *all* tree networks, finding a lower bound for the value of the game by using a new technique we call “pruning” to produce a new tree on which the game is easier to analyze. We also study the game on Eulerian networks, again giving a simple formula for the approximation ratio of a RCPT.

This work lies in the general area of *search games*, on which there is an extensive literature. For good summaries see Alpern and Gal (2003), Garnaev (2000) and Alpern, Fokkink, Gąsieniec, Lindelauf, and Subrahmanian (2014). In addition to the find-and-fetch model of Alpern (2011a), there have been a number of recent extensions to the original model of search games on a network proposed by Gal (1979), for example work of Dagan and Gal (2008) and Alpern, Baston, and Gal (2008) on search games with an arbitrary starting point for the Searcher. Search games in which the Searcher must pay a *search cost* to inspect a node of the network are considered in Baston and Kikuta (2013) and Baston and Kikuta (2015). The expanding search paradigm of Alpern and Lidbetter (2013) models search problems in which the Searcher can move instantaneously back to any point he has already searched, and Alpern and Lidbetter (2015) consider a model in which the Searcher can move at either a slow (searching) speed or a fast (non-searching) speed. There has also been much recent interest in *patrolling games*, which are search games on a network where a Patroller wishes to intercept some terrorist attack, for example Alpern, Morton, and Papadaki (2011), Basilico, Gatti, and Amigoni (2012) and Zoroa, Fernández-Sáez, and Zoroa (2012).

2. Preliminaries

Let Q be a network consisting of a finite connected set of arcs which intersect at nodes, with a distinguished root node O. We may think of a network as an edge weighted multigraph embedded in three-dimensional Euclidean space in such a way that the edges intersect only at nodes of the network. (Three dimensions are the minimum required in general for such an embedding to be possible.) The edge weights correspond to the linear Lebesgue measure μ of the arcs of the network, so that the measure (or *length*) of an arc a is $\mu(a)$. We may consider any subset $A \subset Q$ of points of a network (which may not correspond to a subgraph of the original multigraph). If A is measurable, we write $\mu(A)$ for its length, and we write the total length of Q as $\mu(Q) = \mu$.

A *search strategy* is a unit speed walk $S : \mathbb{R}^+ \to Q$ starting at O, so that $S(0) = O$. More precisely, for any $0 \leq t_1 \leq t_2$, we insist that $d(S(t_1), S(t_2)) \leq t_2 - t_1$, where d is the distance function on pairs of points of Q given by taking the length of the shortest path between them. Since we are thinking of Q as being embedded in Euclidean space, arcs can be traversed in either direction, and the Searcher does not have to finish traversing an arc after she starts, but can turn around and backtrack at any point. A *mixed search strategy*, usually denoted by a lower case letter s is a probabilistic choice of strategies. For a given point H on Q and a given search strategy S, we denote the first time that S reaches H by $T(S, H)$, which we call the *search time*. That is

$$T(S, H) = \min\{t \geq 0 : S(t) = H\}.$$

Note that a hiding place H can be anywhere on the network, including in the interior of an arc. The search time is known to be well defined (see Alpern & Gal, 2003). We consider the problem of determining the mixed strategy that minimizes the expected time to find any point on Q in the worst case. Writing the set of all mixed search strategies as $\mathcal{S}$, the problem is to determine

$$\inf_{s \in \mathcal{S}} \sup_{H \in Q} T(s, H),$$

where $T(s, H)$ is the expected value of the search time of H under s. This can be viewed as a zero-sum game $\Gamma = \Gamma(Q, O)$ between the Searcher, who chooses a search strategy and a malevolent Hider who picks a point on Q. The payoff, which the Searcher seeks to minimize and the Hider to maximize, is the search time. The game is known to have a value, $V = V(Q)$ (see Alpern & Gal, 2003), and the players have optimal strategies, which in general are mixed (randomized). A mixed strategy for the Hider is a distribution over Q and is usually denoted by a lower case letter h. For mixed strategies s and h of the Searcher and Hider, respectively, we write $T(s, h)$ for the expected value of the search time, which we called the *expected search time*.

Gal (1979) showed that if Q is a tree or Eulerian, a RCPT is optimal for the Searcher, as explained in Section 1, so that the value of the game is equal to $\bar{\mu}/2$, where $\bar{\mu}$ is the length of a CPT. In the case of trees, the value is $\bar{\mu}/2 = \mu$ and in the case of Eulerian networks it is $\bar{\mu}/2 = \mu/2$. The complete class of networks for which the RCPT is optimal was later shown by Gal (2001) to be the class of weakly Eulerian networks. We have already informally defined weakly Eulerian networks in Section 1, but we give a formal definition here, which is easily seen to be equivalent. We use