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A stein type lemma for the multivariate generalized hyperbolic distribution

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ABSTRACT

When two variables are bivariate normally distributed, Stein's (1973, 1981) seminal lemma provides a convenient expression for the covariance of the first variable with a function of the second. The lemma has proven to be useful in various disciplines, including statistics, probability, decision theory and finance. In finance, however, asset returns do not always display symmetry but may exhibit skewness. This observation led Adcock (2007, 2010, 2014) to develop Stein's type lemmas for certain multivariate distributions that are consistent with Simaan's (1987, 1993) setting for asset returns. In this paper, we depart from Simaan's setting and develop a new Stein's type lemma in the setting of a mean–variance mixture model for returns. As a particular application, we show that expected utility maximizers select portfolios that are mean–variance–skewness efficient.

1. Motivation

For a bivariate normally distributed random vector $(X, Y)^T$, Stein (1973, 1981)) showed that

$$\mathrm{Cov}[h(X), Y] = \mathrm{Cov}[X, Y]\mathrm{E}[h^{'}(X)], \quad (1)$$

where $h(x)$ is any differentiable function such that $\mathrm{E}[|h'(X)|]$ exists. This result, known as Stein's lemma, has been extended to the multivariate normal setting by Liu (1994) and is useful in various disciplines, including statistics, probability, decision theory and finance. Specifically, when asset returns are multivariate normally distributed, the implication of Stein's lemma is that all expected utility maximizers will select a portfolio that is situated on Markowitz' mean–variance efficient frontier. The lemma also implies Siegel's (1993) formula for the covariance of an arbitrary element of a multivariate normal vector with its minimum element, which has applications in setting up hedging strategies. Brown, DasGupta, Haff, and Strawderman (2006) use the heat equation to establish an identity that is closely related to Stein's lemma and discuss applications in several other research areas, such as graph theory, Bayesian statistics and decision theory. Stein's lemma also leads to an important tool (known as Stein's method) for proving central limit theorems for sums of dependent variables and the approximation of distributions (Barbour & Chen, 2005; Stein, 1986). Some of the implications of Steins lemma were already known (e.g., Siegel's formula[1]), but modern developments extending the lemma to other distributions makes it possible to significantly broaden the scope of these applications. Extending Liu (1994), Landsman and Neslehova's (2008) provide a Stein type lemma that is valid for (multivariate) elliptically distributed variables, which implies that Siegel's formula continues to hold in this setting (Landsman, Vanduffel, & Yao, 2013). Furthermore, optimal portfolios continue to be located on the mean–variance efficient frontier (Adcock, 2010).

In finance, asset returns typically display fatter tails than can be modeled by a normal distribution, and they may also exhibit asymmetry, although evidence of skewness in asset returns is somewhat equivocal. For instance, Arditti and Levy (1975) report that multiperiod returns may be skewed even when single-period returns are not. The opposite conclusion is however reported in both Fogler and Radcliffe (1974) and Lau, Wingender, and Lau (1989). Eberlein and Keller (1995) and Kuechler, Neumann, Soersensen, and Streller (1994) studied daily returns from a sample of German stocks and reported that skewness cannot be ignored. Carr, Geman, Madan, and Yor (2002) investigated the properties of daily asset returns for some U.S. stocks and reported mixed evidence on the existence of skewness and its sign. All these works provide conclusions that are valid for real-world returns. In the context of option pricing, however, so-called risk neutral returns matter, and these are

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[1] Liu (1994) has shown that Siegel's result follows from Stein's lemma in a straightforward way and extended the result to obtain a formula for the covariance between an arbitrary element of a multivariate normal vector with its kth largest element

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typically significantly negatively skewed (Carr et al., 2002). We also refer to the work of Adcock, Eling, and Loperfido (2015) for a review of skewed distributions.

While the elliptical family of distributions can deal with fat tails, it cannot account for asymmetry, and extensions of Stein's lemma to distributions that accommodate this feature are thus of interest. Adcock (2007, 2010, 2014) developed Stein type lemmas for various distributions that allow for asymmetry. Adcock (2007, 2014) also points out the close connection between these distributions and the framework of Simaan (1987, 1993)) for modeling asset returns. In Simaan's framework, the d-dimensional random vector $\mathbf{X}$ (representing asset returns) has the stochastic representation

$$\mathbf{X} = \boldsymbol{\mu} + \boldsymbol{\gamma} W + \mathbf{Z}, \tag{2}$$

where $\mathbf{Z}$ is a d-dimensional elliptically distributed random vector with zero mean, W is a non-negative random variable that is distributed independently of $\mathbf{Z}$ and $\boldsymbol{\mu}, \boldsymbol{\gamma} \in \mathbb{R}^{d\times 1}$. Specifically, $\boldsymbol{\gamma}$ is a vector of skewness parameters and the variable W can be seen as a common (shock) factor[2]. If one considers for $\mathbf{Z}$ the multivariate normal distribution and for W a standard normal distribution that is truncated from below at zero, then $\mathbf{X}$ is said to follow a multivariate skew-normal (MSN) distribution, which was first introduced in Azzalini and Dalla Valle (1996). Allowing the mean of the normal distribution to take any value yields the multivariate extended skew-normal (MESN) distribution, which was independently introduced by Adcock and Shutes (2001) and Arnold and Beaver (2000). Adcock (2007) derives a Stein type lemma for the MESN distribution and applies it to portfolio selection. This author also explores extensions to include more than one skewness shock. Adcock and Shutes (2012) consider a gamma distribution for W and show that a Stein type lemma can be obtained.

In fact, it can be readily shown that in Simaan's setting a version of Stein's lemma can be obtained whenever one considers a distribution for W that makes it possible to conveniently express the quantity $\mathrm{Cov}[h(W), W]$. In such cases it is not necessary for W to be distributed independently of $\mathbf{Z}$. This observation was also made by Adcock (2014, page 394), who obtained a Stein type lemma for the multivariate extended skew Student-t (MEST) distribution for $\mathbf{X}$. In this case, $\mathbf{Z}$ is multivariate Student-t distributed and W has a normal distribution that is truncated from below at zero. The MEST distribution is important, as it offers a parsimonious model to account for salient features of returns, namely fat tails and possible asymmetry.

In this paper, we depart from Simaan's setting and consider a different yet related set-up for modeling asset returns. Specifically, we consider, as model for the d-dimensional random vector $\mathbf{X}$,

$$\mathbf{X} = \boldsymbol{\mu} + \boldsymbol{\gamma} W + \sqrt{W}\mathbf{Z}, \tag{3}$$

where $\boldsymbol{\mu}, \boldsymbol{\gamma} \in \mathbb{R}^{d\times 1}$, $\mathbf{Z}$ specifically denotes a d-dimensional normal random vector with zero mean and covariance matrix $\boldsymbol{\Delta}$ and W is a non-negative random variable. In this setting W can be interpreted as a univariate shock that effectively randomizes the means and covariances of the "normal base model" whereas in Simaan's set-up the base model is elliptical and W solely randomizes its mean vector. Overlap between the two settings appears to occur only under multivariate normality. Many choices for the distribution of the mixing variable W in (3) are possible, but in this paper we consider the case in which W has the so-called generalized Inverse Gaussian (GIG) distribution, which has three parameters and offers a very flexible framework for modeling skewness and (tail) dependence among the assets. In this instance $\mathbf{X}$ has the so-called multivariate generalized hyperbolic (MGH) distribution, which was introduced in the literature by Barndorff-Nielsen (1977, 1978) and Blæsild and Jensen (1981). The study of a Stein type lemma in the MGH setting is interesting for several reasons. First, this setting clearly differs from Simaan's. Furthermore, using MGH distributions to model asset returns (or other data) could lead to better models (Barndorff-Nielsen, 1997; McNeil, Frey, & Embrechts, 2005) and exert a positive impact on the conclusions derived from them (based on Stein's lemma, for instance).

In addition, as we show hereafter, the MGH setting turns out to be tractable and, moreover, is compatible with Lévy models that are routinely used in continuous time finance (in particular, in option pricing models). Finally, parameter estimation and interference has been studied in the literature. In particular, the expectation-maximization (EM) algorithm is useful in this regard; see Dempster, Laird, and Rubin (1977) and McNeil et al. (2005).

Our contributions can be summarized as follows. *First*, we extend Stein's lemma (Stein, 1981) to the MGH setting. Previous extensions were compatible with Simaan's setting, but the MGH setting is different and appears as a natural extension of the multivariate normal distribution, making it possible, for instance, to deal with fat tails and the potential skewness of asset returns. *Second*, our proof of Stein's lemma in the MGH setting relies on a decomposition formula for the quantity $\mathrm{Cov}[h(W), W]$, which we derive under the assumption that W is GIG distributed. In essence, our proof combines this formula for $\mathrm{Cov}[h(W), W]$ with the classic Stein's lemma for multivariate normal distributions. We then notice that in both Simaan's setting and the MGH setting, a version of Stein's lemma can be obtained whenever a distributional choice for W is made that allows one to conveniently decompose the quantity $\mathrm{Cov}[h(W), W]$.

We illustrate this point by extending the Stein type lemma presented in Adcock and Shutes (2012) for the so-called multivariate (skew) normal-gamma distribution (i.e., in (2) W is Gamma distributed) to the general case in which W is GIG distributed. *Third,* we employ Stein's lemma to characterize optimal portfolios in the MGH setting. In particular, we find that all risk-averse expected utility maximizers have their optimal portfolio on a mean–variance–skewness surface. This finding is compatible with the characterization of the optimal portfolio in Birge and Chavez-Bedoya (2015).[3] The results obtained by these authors are valid in the case of an exponential utility function and assuming a GH skewed Student t-distribution for the returns (as a particular case of an MGH distribution); ours, however, are more general in that they hold for all utility functions and all members of the MGH family. In particular, our results also make it possible to compare the (approximate) optimal portfolio derived in the multivariate elliptical framework with the optimal one derived in the MGH framework, as well as to gain insight into the effect of asymmetry on the optimal portfolio.

The remainder of the paper is organized as follows. Section 2 briefly discusses the MGH family. Section 3 generalizes Stein's identity to the MGH family. In Section 4, we present our three-fund separation theorem. Section 5 concludes the paper. Throughout, all statistical quantities mentioned are tacitly assumed to exist.

2. Some facts regarding MGH distributions

In this section, we provide some details regarding MGH distributions, needed in the remainder of the paper; for an extensive treatment we refer to McNeil et al. (2005). We consider first $\mathbf{X} = \boldsymbol{\mu} + \boldsymbol{\gamma} W + \sqrt{W}\mathbf{Z}$, as specified in (3). Provided they exist, the

[2] Adcock and Shutes (2012) interpret W as a shock that represents departures from market efficiency in the sense of Fama (1970).

[3] A similar characterization was obtained by Kwak and Pirvu (2015) in the context of Cumulative Prospect Theory (CPT).