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Constraint propagation using dominance in interval Branch & Bound for nonlinear biobjective optimization

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A B S T R A C T

Constraint propagation has been widely used in nonlinear single-objective optimization inside interval Branch & Bound algorithms as an efficient way to discard infeasible and non-optimal regions of the search space. On the other hand, when considering two objective functions, constraint propagation is uncommon. It has mostly been applied in combinatorial problems inside particular methods. The difficulty is in the exploitation of dominance relations in order to discard the so-called non-Pareto optimal solutions inside a decision domain, which complicates the design of complete and efficient constraint propagation methods exploiting dominance relations.

In this paper, we present an interval Branch & Bound algorithm which integrates *dominance contractors*, constraint propagation mechanisms that exploit an upper bound set using dominance relations. This method discards from the decision space values yielding solutions dominated by some solutions from the upper bound set. The effectiveness of the approach is shown on a sample of benchmark problems.

1. Introduction

Rigorous numerical global optimization aims at finding all the optimal, with respect to some objectives, and feasible, with respect to some constraints, solutions of a nonlinear continuous optimization problem with some numerical guarantees like a prescribed computational precision or solution existence proof. In single-objective optimization, rigorous global methods such as interval *Branch & Bound* (B&B) have been designed and they are well studied in the literature, see e.g. Hansen and Walster (2003), Kearfott (1996b), Neumaier (2004), Van Hentenryck, Michel, and Deville (1997). These methods subdivide the search space into smaller parts which are discarded using bounds on the objective and pruning techniques so as to isolate the portion of the feasible space that contains the global optima. The use of interval analysis allows rigorous computations (e.g., verified linear relaxations) and powerful pruning techniques based on constraint propagation. However, the literature on interval B&B for solving nonlinear biobjective optimization is not proficient. In addition, the recent developments (Fernández & Tóth, 2007; Fernández & Tóth, 2009; Kubica & Woźniak, 2013; Ruetsch, 2005) do not take full benefits of interval analysis, in particular constraint propagation although it has been used within the B&B-like method PICPA (Barichard & Hao, 2003). The difficulty of applying such techniques lies in the exploitation of the dominance relation in the multiobjective case in order to discard *non-optimal* (dominated) solutions of the search space. The method PICPA (Barichard & Hao, 2003) decomposes the objective space, which eases application of constraint propagation but causes overlapping in the decision space.

We propose in this paper an interval B&B algorithm that integrates constraint propagation through the use of *dominance contractors*. These pruning techniques extend the ideas for multiobjective combinatorial optimization presented in Gavanelli (2002), Hartert and Schaus (2014) to nonlinear biobjective continuous optimization. This algorithm generalizes the B&B from Ruetsch (2005), Fernández and Tóth (2009) in which a regular decomposition of the decision space is performed, similar to how it is usually done in the single-objective case. It differs from *inverse* methods (Barichard & Hao, 2003; Kubica & Woźniak, 2013) in which a decomposition of the objective space masters a decomposition in the decision space.

The paper is organized as follows. Sections 2 and 3 introduce the necessary background on nonlinear biobjective optimization and, respectively, on interval analysis and constraint propagation. Our B&B algorithm with dominance contractors is presented in Section 4. Some experiments validating our proposal are

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discussed in Section 5. Eventually, the paper is concluded in Section 6.

2. Nonlinear multiobjective optimization problems

In this section we introduce the terminology and notions in use in multiobjective optimization. Though we consider only biobjective problems in this paper, all the definitions given here apply in the general case and are thus expressed for an arbitrary number m of objectives.

Nonlinear MultiObjective Optimization (NLMOO) consists in optimizing several nonlinear conflicting objectives under nonlinear constraints. Such problems arise in many applications, such as engineering design, the need for a compromise being inherent to the decision process (see, e.g., Ehrgott, 2005; Miettinen, 1999). A NLMOO problem can be written as follows:

$$\begin{bmatrix} \min & f(x) \\ \text{s.t.} & g(x) \leq 0 \\ & h(x) = 0 \end{bmatrix} \tag{1}$$

with $x \in \mathbb{R}^n$ the decision variables, $f : \mathbb{R}^n \to \mathbb{R}^m$ the objective functions, $g : \mathbb{R}^n \to \mathbb{R}^p$ the inequality constraints and $h : \mathbb{R}^n \to \mathbb{R}^q$ the equality constraints. The feasible region $\mathcal{X}$ is the set of decision vectors that satisfy all the constraints, i.e., $\mathcal{X} := \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0\}$. Its image $\mathcal{Y} = f(\mathcal{X})$ in the objective space is called the feasible objective region. In this paper, we consider objective and constraints that are continuously differentiable[1].

Because the objective functions are conflicting, all feasible objective vectors cannot be compared. Still, if one such vector $y \in \mathcal{Y}$ is better according to all the objective functions than another one $y' \in \mathcal{Y}$, y is obviously more desirable than y'. This is formalized by the notion of dominance.

Definition 1 (Dominance relations)**.** Let y and y' be two vectors in $\mathbb{R}^m$. The following notations are used:

(i) $y < y' \equiv y_i < y'_i \quad \forall i = 1, \ldots, m$ (y strictly dominates y')
(ii) $y \lneq y' \equiv \quad y_i \leq y'_i \quad \forall i = 1, \ldots, m,$ and $y \neq y'$ (y dominates y')
(iii) $y \leq y' \quad \equiv \quad y_i \leq y'_i \quad \forall i = 1, \ldots, m$ (y weakly dominates y')

In a posteriori decision making (i.e., without preferences inducing an aggregation of the objectives), solving problem (1) requires computing its set of Pareto optimal solutions, i.e., optimal tradeoffs between the objectives.

Definition 2 (Nondominance, Pareto optimality)**.** Consider a feasible objective vector $y \in \mathcal{Y}$. It is a *nondominated* (resp. *weakly nondominated*) vector of $\mathcal{Y}$ if there is no other $y' \in \mathcal{Y}$ such that $y' \lneq y$ (resp. $y' < y$). The set of nondominated (resp. weakly nondominated) vectors is denoted $\mathcal{Y}^*$ (resp. $\mathcal{Y}^*_W$).

A feasible solution $x \in \mathcal{X}$ is *Pareto optimal* (resp. *weakly Pareto optimal*) if $f(x)$ is nondominated (resp. weakly nondominated). The set of Pareto optimal (resp. weakly Pareto optimal) solutions is denoted by $\mathcal{X}^*$ (resp. $\mathcal{X}^*_W$).

As the objectives and constraints are nonlinear (non-convex), locally Pareto optimal solutions may exist.

Definition 3 (Local optimality)**.** A solution $x \in \mathcal{X}$ is *locally Pareto optimal* if there exists $\delta > 0$ such that x is Pareto optimal in the ball $\mathcal{B}(x, \delta) \cap \mathcal{X}$.

In the convex case, all locally Pareto optimal solutions are globally Pareto optimal (Miettinen, 1999, Theorem 2.2.3) and can be found using local approaches, e.g., as a set of Pareto optimal solutions with images well spread upon the nondominated set. Oppositely, the non-convex case requires global search methods like evolutionary algorithms (Coello, Lamont, & Van Veldhuizen, 2006), swarm algorithms (Reyes-Sierra & Coello, 2006), or interval B&B (Fernández & Tóth, 2009; Kubica & Woźniak, 2013; Ruetsch, 2005).

Computing all the globally Pareto optimal solutions via interval B&B requires bounding the subproblems issued from the subdivision of the search space. Contrarily to the single-objective case, bounding in multiobjective optimization is not straightforward: the bounds must enclose a whole set of nondominated vectors[2]. Usually, the ideal y^I and nadir y^N (or anti-ideal y^A) points are used to bound, respectively below and above, the nondominated set $\mathcal{Y}^*$:

$$y^I_i = \min\nolimits_{x \in \mathcal{X}} f_i(x) = \min\nolimits_{y \in \mathcal{Y}} y_i, \ i = 1, \ldots, m \tag{2}$$

$$y^N_i = \max\nolimits_{x \in \mathcal{X}^*} f_i(x) = \max\nolimits_{y \in \mathcal{Y}^*} y_i, \ i = 1, \ldots, m \tag{3}$$

$$y^A_i = \max\nolimits_{x \in \mathcal{X}} f_i(x) = \max\nolimits_{y \in \mathcal{Y}} y_i, \ i = 1, \ldots, m \tag{4}$$

As seen on Fig. 1, the ideal and nadir bound the nondominated set $\mathcal{Y}^*$: all nondominated points are dominated by the ideal and dominate the nadir, while all feasible objective points dominate the anti-ideal. Those particular points can be "easily" computed in the biobjective case[3] provided solutions of single-objective versions of Problem (1) are known (or can be efficiently obtained). On the other hand, as they are single points, they do not provide good bounds, capturing only poorly the shape of the nondominated set. In order to obtain a more accurate bounding, dominance-free bounding sets have been introduced in Ehrgott and Gandibleux (2007).

Definition 4 (Dominance-free set)**.** A set E of vectors in $\mathbb{R}^m$ is *dominance-free* if there is no $y, y' \in E$ such that y dominates y'.

Intuitively, a dominance free set can serve as a lower (resp. upper) bound if it is "below" (resp. "above") the set of Pareto optimal solutions to the problem. A formal definition follows. The right hand side of Fig. 1 depicts one lower and one upper bound set.

Definition 5 (Bound sets)**.** Consider Problem (1) and let $\mathcal{Y}_L \subset \mathbb{R}^m$ be a dominance-free set. This set is a *lower bound set* of $\mathcal{Y}^*$ if it satisfies:

$$\mathcal{Y}^* \subseteq \{y : \exists y' \in \mathcal{Y}_L, y \geq y'\}$$

Similarly, let $\mathcal{Y}_U \subset \mathbb{R}^m$ be a dominance-free set. This set is an *upper bound set* of $\mathcal{Y}^*$ if it satisfies:

$$\mathcal{Y}^* \subseteq \mathbb{R}^m \setminus \{y : \exists y' \in \mathcal{Y}_U, y > y' \}$$

Given this definition, any dominance-free set of feasible objective vectors form a global upper bound set of Problem (1), e.g., the black points in Fig. 1. Note also that bound sets can be used to locally bound the Pareto optimal solutions in sub-regions of the search space.

3. Interval analysis

Interval analysis (IA) is a modern branch of numerical analysis born in the 1960's (Moore, 1966). It replaces computations with real numbers by computations with intervals of real numbers, providing a framework for handling uncertainties and verified computations. It is a powerful tool for dealing reliably with any problems implying real-valued variables such as numerical constraint satisfaction and nonlinear optimization (Jaulin, Kieffer, Didrit, & Walter, 2001; Kearfott, 1996a,b; Neumaier, 1991).

[1] Though constraint propagation could apply to evaluable only (blackbox) functions, its effectiveness is reduced in this case.

[2] Assuming this set is actually bounded.

[3] Weakly nondominated points increase the difficulty of computing y^N.