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Decision Support

Comonotonic approximation to periodic investment problems under stochastic drift

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A B S T R A C T

We investigate periodic investment problems under a Black–Scholes market with stochastic drift. The decision maker invests a series of positive amounts at finitely predetermined time spots, to maximize the expected terminal wealth while controlling its downside risk as measured by the Condition Value at Risk (CVaR). It turns out that the increment for unit wealth on the whole path can be divided into two parts: the increment corresponding to the stochastic drift and that corresponding to the Brownian Motion. A comonotonic approximation is proposed for the second part, and an upper bound is provided for the CVaR of the first part, which construct together a closed-form approximation of the terminal wealth under the risk measure of CVaR. We further decompose the problem into a sequence of sub-problems whose optimal solutions are explicit and follow fractional Kelly Strategy. Numerical and empirical results illustrate the performance of our methodology.

1. Introduction

Portfolio selection refers to determining the best allocation of wealth among various securities (e.g., risky and risk-free assets) over a given horizon. Markowitz (1952, 1956) introduced the mean-variance portfolio selection model and provided the foundation for modern finance theory, inspiring a substantial number of extensions and applications. The pioneering work of Merton (1972) created analytical portfolio policies and the mean-variance efficient frontier for static mean-variance portfolio selection problems. Moreover, Li and Ng (2000) and Zhou and Li (2000) extended the formulation to multi-period and continuous time settings, respectively. Cui, Gao, Li, and Li (2014) further advanced this problem by characterizing the structure of optimal portfolio policies. Most literature on dynamic portfolio selection, e.g., surveyed by Detemple (2014), Abdelaziz, Aouni, and El Feyedh (2007) and Kolm, Tütüncü, and Fabozzi (2014) among others, focused on self-financing settings, i.e., the investor cannot inject extra capital into or withdraw cash flow from the portfolio during intermediate periods. As a generalization of classical portfolio selection problems, the periodic investment problem, considered in this paper, assumes that the decision maker invests a given series of positive amounts α_i, at predetermined time spots, $1 \le i \le N$, such that the expectation of his/her terminal wealth at time N would be maximized, while keeping the downside risk as measured by the Conditional Value at Risk (CVaR) under control. Particularly, we assume that prices of risky assets satisfy the Black–Scholes model (Black & Scholes, 1973) with stochastic drift.

Dhaene, Vanduffel, Goovaerts, Kaas, and Vyncke (2005) investigated the periodic investment problem, where risky assets follow the Black–Scholes model with constant drift. However, as indicated in Chopra and Ziemba (1993), specifying the expected rate of return (or the drift) of the risky asset based on statistical forecasting procedures may suffer from large estimation errors, which would significantly damage optimal portfolio selection decisions. In light of this observation, MacLean, Zhao, and Ziemba (2006) adopted the Black–Scholes model with stochastic drift to investigate dynamic portfolio selection problems with process control. They developed fractional Kelly-strategy type optimal solutions as qX^*, where X^* denotes the well-known Kelly portfolio strategy (or the so called optimal growth portfolio in literature) proposed in Kelly (1956) and fraction q controls the downside risk. It is worth noting that, MacLean et al. (2006) applied a Bayesian framework in their model specification to update forecasts of the stochastic drift whose prior distribution is assumed to be normal. In contrast, we adopt a distribution-free setting for the stochastic drifts of risky assets, only specifying their means and covariance

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matrix. This distribution-free setting is common in the literature. For instance, without strong assumptions about the distribution, Bertsimas and Pachamanova (2008) suggested different robust formulations of the multi-period portfolio management problem with transaction costs. They showed that a robust polyhedral optimization, in particular, could enhance the performance of single-period and deterministic multi-period portfolio optimization methods. However, their robust approach is based on the uncertainty set for asset return rather than the stochastic drift under the Black–Scholes model.

From a mathematical perspective, risk management is a procedure for shaping a risk distribution. Popular risk measures are VaR and CVaR. Sarykalin, Serraino, and Uryasev (2008) provided introductions to these risk measures with their applications. Inspired by the risk management procedures proposed in Sarykalin et al. (2008), the periodic investment problem in this paper is formulated as an optimization problem which maximizes the expected return of terminal wealth, while controlling its downside risk as measured by CVaR. We develop an approach based on the comonotonic approximation methodology to tackle this problem. The concept of comonotonicity is a powerful tool in actuarial science and finance (see Dhaene, Denuit, Goovaerts, Kaas, & Vyncke, 2002a; 2002b) for a comprehensive summary of its theory and applications, respectively). For example, Dhaene et al. (2005), Dhaene and Goovaerts (1996) and Kaas, Dhaene, and Goovaerts (2000) developed comonotonic approximations for classical portfolio selection problems (without stochastic drift in the dynamics of risky assets) within the family of constant mix strategies, while Pagnoncelli and Vanduffel (2012) applied such approximations to provisioning problems related to periodic investment.

In this paper, we obtain closed-form comonotonic approximations to the periodic investment problems under CVaR risk control, with an assumption of geometric Brownian Motion with stochastic drift. The increment for unit wealth on the entire path can be divided into two parts: the increment corresponding to the stochastic drift and that corresponding to the Brownian Motion. A comonotonic approximation is proposed for the second part and an upper bound is provided for the CVaR of the first part, respectively, which together construct a closed-form approximation to the terminal wealth under the risk measure of CVaR. We further decompose the original multi-period problem into a sequence of single-period ones whose optimal solutions are explicit and follow fractional Kelly Strategy. In summary, our contributions and their significance are as follows:

(1) We study the periodic investment problem under the setting of a Black–Scholes model with stochastic drift characterized by a distribution-free framework. The distribution-free setting under the Black–Scholes model is new to the literature of periodic investment problems.
(2) Based on our new setting, different from MacLean et al. (2006) and Dhaene et al. (2005), we divide the terminal wealth under CVaR into one part corresponding to the Brownian Motion and the other corresponding to the stochastic drift.
(3) Based on the definition for CVaR in Sarykalin et al. (2008), we derive closed-form upper bounds for CVaR over the terminal wealth corresponding to the stochastic drift and Brownian Motion, respectively.
(4) We decompose the terminal wealth problem into a series of sub-problems within the constant mix strategy, whose optimal solutions follow the fractional Kelly Strategy. Compared to MacLean et al. (2006), our proposed solution technique solves CVaR over stochastic drift within the distribution-free setting.
(5) Based on the optimal constant mix strategy, we design a dynamic control strategy in a rolling horizon manner for the periodic investment problem.

The remainder of the paper is organized as follows. The terminology and preliminaries are introduced in Section 2. We then set up the terminal wealth problem in Section 3. As previously mentioned, the increment for the portfolio can be divided into the one corresponding to the stochastic drift and the other corresponding to the Brownian Motion. In Section 4, an upper bound for the CVaR of the first part is developed and CVaR for the second part is calculated. Furthermore, based on the closed form of terminal wealth under CVaR, the solution technique for the terminal wealth problem is provided by decomposing it into subproblems. For each subproblem, the closed form of optimal portfolio selection is proved to be a fractional Kelly Strategy. Subsequently, the dynamic control strategy for the periodic investment problem is developed based on the constant mix strategy by a rolling horizon manner in Section 5. The efficiency of our approximation for terminal wealth is tested via numerical experiments and the performance of proposed periodic investment strategy is shown by empirical experiments in Section 6. Section 7 concludes the paper and proposes future research direction. All proofs are presented in Appendix A.

2. Preliminary settings

2.1. Periodic investment problems

Consider a multi-period discrete-time horizon with N periods and the investor has endowments $\alpha_k, k = 1, 2, \ldots, N$ at the beginning of the k-th period. Suppose the investor can allocate his/her wealth in a basket of $m+1$ assets, i.e., m risky assets and one risk-free asset. Let $X(t) = (x_1(t), x_2(t), \ldots, x_m(t))$ be the vector of proportions invested in each risky asset, while $1 - \sum_{i=1}^{m} x_i(t)$ is invested in the risk-free asset. A negative proportion invested in the risk-free asset is allowed. The investor can only rebalance his/her portfolio at the beginning of each period. We further assume the following Black–Scholes model with stochastic drifts for the prices of risky assets:

$$\frac{dP_i(t)}{P_i(t)} = \tilde{\mu}_i dt + \sigma_i dB_i(t), \qquad P_i(0) > 0, \qquad 1 \le i \le N,$$

and $P_0(t) = P_0(0)e^{rt}$ for the risk-free asset, where $B_i(t)$, $1 \le i \le N$ are (correlated) standard Brownian Motions with $Cov(B_i(t), B_j(t+s)) = \rho_{ij} t$ for $1 \le i, j \le N$ and $t, s \ge 0$. Note that the correlation coefficients ρ_{ij}, $1 \le i, j \le N$ are constants and the drifts $\tilde{\mu}_i$ for $1 \le i \le N$ are specified as follows. Similar to MacLean et al. (2006), for $1 \le i \le N$, the drift $\tilde{\mu}_i$ depends on common factors F_l, $l = 1, 2, \ldots, s$, as:

$$\tilde{\mu}_i = \bar{\mu}_i + \sum_{l=1}^{s} \lambda_{il} F_l, \tag{1}$$

where $\bar{\mu}_i$ is a constant and F_l are normalized independent market factors (with zero mean and unit variance). Unlike the Bayesian Framework in MacLean et al. (2006), we do not make strong assumptions on distributions of $\tilde{\mu}_i$ and F_l. It is only assumed that the mean and co-variance matrix for $\tilde{\mu}_i$, $i = 1, 2, \ldots, m$, are known. From (1), since F_l for $l = 1, 2, \ldots, s$ are independent, the element γ_{ij} in covariance matrix $\Sigma' := (\gamma_{ij})_{m \times m}$ of $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_m)$ are determined by

$$\gamma_{ij} = \sum_{l=1}^{s} \lambda_{il} \lambda_{jl}. \tag{2}$$