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Continuous Optimization

Cell-and-bound algorithm for chance constrained programs with discrete distributions

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ABSTRACT

Chance constrained programing (CCP) is often encountered in real-world applications when there is uncertainty in the data and parameters. We consider in this paper a special case of CCP with finite discrete distributions. We propose a novel approach for solving CCP. The methodology is based on the connection between CCP and arrangement of hyperplanes. By involving cell enumeration methods for an arrangement of hyperplanes in discrete geometry, we develop a cell-and-bound algorithm to identify an exact solution to CCP, which is much more efficient than branch-and-bound algorithms especially in the worst case. Furthermore, based on the cell-and-bound algorithm, a new polynomial solvable subclass of CCP is discovered. We also find that the probabilistic version of the classical transportation problem is polynomially solvable when the number of customers is fixed. We report preliminary computational results to demonstrate the effectiveness of our algorithm.

1. Introduction

We consider in this paper the following individual chance-constrained program:

$$
\begin{aligned}
(\mathrm{IC^2P}) \quad & \min f(x) \\
& \text{s.t.}\, \mathbb{P}(\xi^T Bx \geq \rho) \geq 1-\alpha, \qquad (1)\\
& x \in \mathcal{O},
\end{aligned}
$$

where $f(x)$ is a convex function of x, $\mathbb{P}(\cdot)$ denotes the probability, ξ is a random vector taking values in $\Re^m$, B is an $m \times d$ matrix, $\rho \in R$, $\alpha \in (0, 1)$ is a prescribed risk level which is given by the decision maker, typically near zero, e.g., $\alpha = 0.01$ or $\alpha = 0.05$, and $\mathcal{O}$ is a convex compact set. We will show in Section 2.2 that our cell-and-bound algorithm can be generalized parallel to the following chance-constrained program (CCP):

$$
\begin{aligned}
(\mathrm{C^2P}) \quad & \min\ f(x) \\
& \text{s.t.}\ \mathbb{P}(\Gamma x \geq \varepsilon) \geq 1-\alpha, \qquad (2)\\
& x \in \mathcal{O},
\end{aligned}
$$

where Γ is an $m \times d$ random matrix and ε is a random vector taking values in $\Re^m$. Constraint (2) is called a *chance constraint*. Especially, when $m = 1$, i.e., constraint (1) is called an *individual chance constraint*. Constraint (2) is also known as a *joint probabilistic constraint* where we require all constraints be satisfied simultaneously, rather than having each row satisfied independently with different probabilities. It has been shown in Luedtke, Ahmed, and Nemhauser (2010) that problem (C^{2}P) is NP-hard even when the matrix Γ is deterministic. Thus, problem (C^{2}P) is NP-hard in general. When only the left-hand side of the probabilistic constraint is random, i.e., ε is deterministic and matrix Γ is random, problem (C^{2}P) reduces to chance-constrained problems with a random technology matrix (CCRTM). Similarly, when only the right-hand side of the probabilistic constraint is random, i.e., ε is random and matrix Γ is deterministic, problem (C^{2}P) reduces to chance-constrained problems with random right-hand sides (CCRRH).

Many problems in various areas can be formulated as (C^{2}P). Lejeune (2012) reviewed a series of applications, for example, portfolio selection of Gaivoronski and Pflug (2005), multistage supply chain management of Lejeune and Ruszczynski (2007), military logistics problem of Kress, Penn, and Polukarov (2007), managing stochastic pollution of waters of Gren, (2008) and so on. Problem (C^{2}P) was first introduced by Charnes, Cooper, and Symonds (1958), Miller and Wagner (1965) and Prékopa (1970). It is well known that when the random input has a joint normal distribution, problem (C^{2}P) can be reduced to a convex problem and thus can be solved efficiently via convex programing techniques (see Deák, 2000; Szántai, 2000). Unfortunately, problem (C^{2}P) is

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generally difficult to solve due to two major reasons. Firstly, the feasible region together with the probabilistic constraint is usually not convex, although the set $\mathcal{O}$ is convex. Secondly, the probability $\mathbb{P}(\Gamma x \geq \varepsilon)$ is typically difficult to compute since multi-dimensional integration is required.

For the case when the random input is continuous with known probability distributions, different approaches have been proposed in the literature to overcome the difficulty of the nonconvexity of the feasible set defined by the chance constraint. There are excellent surveys on probabilistic constrained problems in Prékopa (2003) and Shapiro, Dentcheva, and Ruszczyński (2009) and our reviews here are mainly inspired by Lejeune (2012). Problem CCRTM was first studied by Kataoka (1963) and Van de Panne and Popp (1963), where they studied convexity of the individual chance constraint $\mathbb{P}(\xi^T x \leq d) \geq p$ with ξ being random. Later, Henrion (2007) extended the convexity results of Kataoka (1963) and Van de Panne and Popp (1963) to a broader class of distributions and to more general functions of the decision vector. More precisely, Henrion (2007) studied the convexity properties of the chance constraint $\mathbb{P}(\xi^T h(x) \leq d) \geq p$ with ξ being random. For the case of the multirow chance constraint, Prékopa (1974) proved the quasi-concavity of the function $G(x) = \mathbb{P}(\Gamma x \leq \beta)$ under some conditions, where only the matrix Γ was random. For problem CCRRH, (Prékopa, 1973) showed that the feasible set of problem (C^2P) was convex if the random variable ε in constraint (2) was continuously distributed with logarithmically concave probability density functions. Together with the assumption of the convexity of function $f(x)$, problem (C^2P) was hence a convex program and the solution to (C^2P) could be found via convex programing techniques. Henrion and Strugarek (2008) studied the convexity properties of chance constraints $\mathbb{P}(h_i(x) \geq \zeta_i, i = 1, \ldots, r) \geq p$ with ζ being random. For more general chance constraint, i.e., $\mathbb{P}(F(x, \xi) \leq 0) \geq 1 - \alpha$, a convex safe (conservative) approximation was proposed in Nemirovski and Shapiro (2006) to build computationally tractable convex inner approximations.

The case when the random variables are discretely distributed is extensively studied, such as Lejeune and Ruszczynski (2007), Luedtke and Ahmed (2008), Kücükyavuz (2012) and so on. Discrete distributions arise frequently in sample approximation of the underlying distribution. If the possible values for random input, which is also referred as *scenarios*, are generated by taking a Monte Carlo sample from a general distribution, the resulting problem can be viewed as an approximation of the problem with the general distribution. We can find feasible solutions and lower bounds for the original problem by such sample approximation methods. Furthermore, the required sample size is polynomial in $1/\alpha$ (Luedtke & Ahmed, 2008). Related results can be found in Shapiro and Homem-de Mello (2000), Atlason, Epelman, and Henderson (2004), and Henrion and Römisch (2004). By associating a binary variable with each scenario, problem (C^2P) with discrete distributions can be reduced to a mixed-integer programing (MIP) reformulation. In the general case, the number of binary variables usually grows linearly with the number of scenarios. In addition, in scenario approximation approaches, the number of scenarios is $O(1/\alpha)$ and the risk level α in constraint (2) is typically near zero, e.g., $\alpha = 0.01$ or $\alpha = 0.05$. Therefore, the size of the resulted MIP reformulation of (C^2P) is usually much larger than the original problem (C^2P). Hence the difficulty of solving the resulted MIP reformulation will be increased. Meanwhile, the MIP reformulation is commonly solved by the MIP solvers in the framework of branch-and-bound. The efficiency of the branch-and-bound methods largely depends on the tightness of the lower bounds obtained by continuous relaxation. However, numerical tests, such as Zheng, Sun, Li, and Cui (2012) and Luedtke et al. (2010), suggest that the continuous relaxation of the standard MIP reformulation often provides poor lower bounds. Thus, the MIP solvers in the framework of branch-and-bound can not solve the MIP reformulation of (C^2P) efficiently.

Based on the above discussion, for problem CCP with discrete distributions, there were many researches on the improved MIP reformulation in the sense that the improved MIP reformulation could be solved much more efficiently than the standard MIP reformulation. For CCRRH, Ruszczyński (2002) developed cutting planes and integrated them into a branch-and-cut algorithm to solve its MIP reformulation. Luedtke et al. (2010) proposed two strengthened MIP reformulations by deriving strong valid inequalities. Recently, Lejeune (2012) proposed a novel modeling and solution method for CCRRH with a linear objective and linear constraints. It was the first time that problem CCP was solved by employing the techniques from the pattern recognition field. The crucial aspect of the method in Lejeune (2012) was that the method could find the exact solution of CCP fast even though the number of scenarios were extremely large. For CCRTM, Zheng et al. (2012) proposed an improved MIP reformulation for CCRTM with Value-at-Risk constraint $\mathbb{P}(\xi^T Bx \geq R) \geq 1 - \alpha$, in the sense that the continuous relaxation of the improved MIP reformulation was tighter than or at least as tight as that of the standard MIP reformulation. Beraldi, Bruni, and Guerriero (2010) proposed a tailor-made branch-and-bound approach to solve the MIP reformulation of the probabilistic set covering problem. Kogan and Lejeune (2014) developed a new modeling and solution method for CCRTM with a linear objective, linear constraints and discretely distributed matrix Γ. They derived three new deterministic formulations and a set of strengthening valid inequalities for their studied problems in Kogan and Lejeune (2014). An impressive feature of the integer reformulations of Kogan and Lejeune (2014) was that the number of integer variables was independent of the number of scenarios. For a more complex and widely applicable class of chance constraints, i.e., $\mathbb{P}(\sum_{j=(j_1,j_2)\in J} s_{ij} x_{j1} x_{j2} \xi_j \leq d_i, i \in I) \geq 1 - p$ with ξ being random, Lejeune and Margot (2016) proposed a new and systematic reformulation and algorithmic framework to solve this class of problems. Such complex stochastic quadratic inequalities were first studied by Lejeune and Margot (2016). The key distinguishing feature of the proposed method was that the number of binary variables did not grow linearly with the number of scenarios.

In the rest of this paper, we focus on problem (IC^2P) with following finite discrete distribution assumption:

Assumption 1. The random vector ξ has a finite distribution, i.e., there are only finite realizations (scenarios) $\xi^1, \ldots, \xi^N \in \Re^m$ of ξ with probability $p_1, \ldots p_N$ and $\sum_{j=1}^N p_j = 1$.

We propose in this paper a novel approach for solving chance constrained problems under Assumption 1. Our method is based on recognition that whether or not a scenario ξ^j satisfies $(\xi^j)^T Bx \geq \rho$ is equivalent to whether or not the vector x lies on the positive side of hyperplane $\{x \mid (\xi^j)^T Bx = \rho\}$. By involving the cell enumeration method for an arrangement of hyperplanes in discrete geometry, we develop a cell-and-bound algorithm to identify the optimal solution to (C^2P). Our new algorithm is more efficient than continuous relaxation based branch-and-bound algorithms in the worst case. Based on the cell-and-bound algorithm, we discover a new polynomial subclass of problem (C^2P). Furthermore, Our new approach provides a promising platform for solving (C^2P) efficiently. The main contributions of this work are as follows:

- Our new approach opens the interesting connection between probabilistically constrained programing and arrangement of hyperplanes in discrete geometry. Our new approach provides a promising platform for solving (C^2P) efficiently.
- By involving cell enumeration approach, we propose a novel approach, named cell-and-bound algorithm, for solving problem (C^2P) with finite discrete distributions.