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Continuous Optimization

Dual Dynamic Programing with cut selection: Convergence proof and numerical experiments

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ABSTRACT

We consider convex optimization problems formulated using dynamic programing equations. Such problems can be solved using the Dual Dynamic Programing algorithm combined with the Level 1 cut selection strategy or the Territory algorithm to select the most relevant Benders cuts. We propose a limited memory variant of Level 1 and show the convergence of DDP combined with the Territory algorithm, Level 1 or its variant for nonlinear optimization problems. In the special case of linear programs, we show convergence in a finite number of iterations. Numerical simulations illustrate the interest of our variant and show that it can be much quicker than a simplex algorithm on some large instances of portfolio selection and inventory problems.

1. Introduction

Dual Dynamic Programing (DDP) is a decomposition algorithm to solve some convex optimization problems. The algorithm computes lower approximations of the cost-to-go functions expressed as a supremum of affine functions called optimality cuts. Typically, at each iteration, a fixed number of cuts is added for each cost-to-go function. It is the deterministic counterpart of the Stochastic Dual Dynamic Programing (SDDP) algorithm pioneered by Pereira and Pinto (1991). SDDP is still studied and has been the object of several recent improvements and extensions (Guigues, 2014b; Guigues & Römisch, 2012a, 2012b; Kozmik & Morton, 2015; Pfeiffer, Apparigliato, & Auchapt, 2012; Philpott & de Matos, 2012; Philpott, de Matos, & Finardi, 2012; Shapiro, 2011; Shapiro, Tekaya, da Costa, & Soares, 2013). In particular, these last three references discuss strategies for selecting the most relevant optimality cuts which can be applied to DDP. In stochastic optimization, the problem of cut selection for lower approximations of the cost-to-go functions associated to each node of the scenario tree was discussed for the first time in Ruszczyński (1993) where only the active cuts are selected. Pruning strategies of basis (quadratic) functions have been proposed in Gaubert, McEneaney, and Qu (2011) and McEneaney, Deshpande, and Gaubert (2008) for max-plus based approximation methods which, similarly to SDDP, approximate the cost-to-go functions of a nonlinear optimal control problem by a supremum of basis functions. More precisely, in Gaubert et al. (2011), a fixed number of cuts is pruned and cut selection is done solving a combinatorial optimization problem. For SDDP, in Shapiro et al. (2013) it is suggested at some iterations to eliminate redundant cuts (a cut is redundant if it is never active in describing the lower approximate cost-to-go function). This procedure is called *test of usefulness* in Pfeiffer et al. (2012). This requires solving at each stage as many linear programs as there are cuts. In Pfeiffer et al. (2012) and Philpott et al. (2012), only the cuts that have the largest value for at least one of the trial points computed are considered relevant, see Section 4 for details. This strategy is called the *Territory algorithm* in Pfeiffer et al. (2012) and Level 1 cut selection in Philpott et al. (2012). It was presented for the first time in 2007 at the ROADEF congress by David Game and Guillaume Le Roy (GDF-Suez), see Pfeiffer et al. (2012). However, a difference between Pfeiffer et al. (2012) and Philpott et al. (2012) is that in Pfeiffer et al. (2012) the nonrelevant cuts are pruned whereas in Philpott et al. (2012) all computed cuts are stored and the relevant cuts are selected from this set of cuts.

In this context the contributions of this paper are as follows. We propose a limited memory variant of Level 1. We study the convergence of DDP combined with a class of cut selection strategies that satisfy an assumption (Assumption (H2), see Section 4.2) satisfied by the Territory algorithm, the Level 1 cut selection strategy, and its variant. In particular, the analysis applies to (i) mixed cut selection strategies that use the Territory algorithm, Level 1, or its variant to select a first set of cuts and then apply the test of usefulness to these cuts and to (ii) the Level H cut selection strategy from Philpott et al. (2012) that keeps at each trial point the H cuts having the largest values. In the case when the problem is linear, we additionally show convergence in a finite

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number of iterations. Numerical simulations show the interest of the proposed limited memory variant of Level 1 and show that it can be much more efficient than a simplex algorithm on some instances of portfolio selection and inventory problems.

The outline of the study is as follows. Section 2 recalls from Guigues (2014a) a formula for the subdifferential of the value function of a convex optimization problem. It is useful for the implementation and convergence analysis of DDP applied to convex nonlinear problems. Section 3 describes the class of problems considered and assumptions. Section 4.1 recalls the DDP algorithm while Section 4.2 recalls Level 1 cut selection, the Territory algorithm and describes the limited memory variant we propose. Section 5 studies the convergence of DDP with cut selection applied to nonlinear problems while Section 6 studies the convergence for linear programs. Numerical simulations are reported in Section 7.

We use the following notation and terminology:

- The usual scalar product in $\mathbb{R}^n$ is denoted by $\langle x, y\rangle = x^\top y$ for $x, y \in \mathbb{R}^n$.
- ri(A) is the relative interior of set A.
- $\mathbb{B}_n$ is the closed unit ball in $\mathbb{R}^n$.
- dom(f) is the domain of function f.
- $|I|$ is the cardinality of the set I.
- $\mathbb{N}^*$ is the set of positive integers.

2. Formula for the subdifferential of the value function of a convex optimization problem

We recall from Guigues (2014a) a formula for the subdifferential of the value function of a convex optimization problem. It plays a central role in the implementation and convergence analysis of DDP method applied to convex problems and will be used in the sequel.

Let $\mathcal{Q} : X \to \overline{\mathbb{R}}$, be the value function given by

$$\mathcal{Q}(x) = \begin{cases} \inf_{y\in\mathbb{R}^n} \; f(x, y) \\ y \in S(x) := \{y \in Y : \; Ax + By = b, \; g(x, y) \le 0\}. \end{cases} \tag{2.1}$$

Here, A and B are matrices of appropriate dimensions, and $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ are nonempty, compact, and convex sets. Denoting by

$$X^\varepsilon := X + \varepsilon\mathbb{B}_m \tag{2.2}$$

the ε-fattening of the set X, we make the following assumption (H):

(1) $f : \mathbb{R}^m\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is lower semicontinuous, proper, and convex.
(2) For $i = 1, \ldots, p$, the ith component of function $g(x, y)$ is a convex lower semicontinuous function $g_i : \mathbb{R}^m\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$.
(3) There exists $\varepsilon > 0$ such that $X^\varepsilon \times Y \subset \text{dom}(f)$.

Consider the dual problem

$$\sup_{(\lambda,\mu)\in\mathbb{R}^q\times\mathbb{R}^p_+} \theta_x(\lambda, \mu) \tag{2.3}$$

for the dual function

$$\theta_x(\lambda, \mu) = \inf_{y\in Y} \; f(x, y) + \lambda^\top(Ax + By - b) + \mu^\top g(x, y).$$

We denote by $\Lambda(x)$ the set of optimal solutions of the dual problem (2.3) and we use the notation

$$\text{Sol}(x) := \{y \in S(x) : f(x, y) = \mathcal{Q}(x)\}$$

to indicate the solution set to (2.1).

It is well known that under Assumption (H), $\mathcal{Q}$ is convex. The description of the subdifferential of $\mathcal{Q}$ is given in the following lemma:

Lemma 2.1 (Lemma 2.1 in Guigues, 2014a)**.** *Consider the value function $\mathcal{Q}$ given by (2.1) and take $x_0 \in X$ such that $S(x_0) \neq \emptyset$. Let Assumption (H) hold and assume the Slater-type constraint qualification condition:*

$$\textit{there exists } (\bar{x}, \bar{y}) \in X\times ri(Y) \text{ such that } A\bar{x} + B\bar{y} = b \text{ and } (\bar{x}, \bar{y}) \in ri(\{g \le 0\}).$$

Then $s \in \partial\mathcal{Q}(x_0)$ if and only if

$$(s, 0) \in \partial f(x_0, y_0) + \{[A^\top; B^\top]\lambda : \; \lambda \in \mathbb{R}^q\} + \left\{ \sum_{i\in I(x_0,y_0)} \mu_i \partial g_i(x_0, y_0) : \; \mu_i \ge 0 \right\} + \{0\}\times\mathcal{N}_Y(y_0), \tag{2.4}$$

where y_0 is any element in the solution set Sol(x_0) and with

$$I(x_0, y_0) = \{i \in \{1, \ldots, p\} : \; g_i(x_0, y_0) = 0\}.$$

In particular, if f and g are differentiable, then

$$\partial\mathcal{Q}(x_0) = \left\{ \nabla_x f(x_0, y_0) + A^\top\lambda + \sum_{i\in I(x_0,y_0)} \mu_i \nabla_x g_i(x_0, y_0) : \; (\lambda, \mu) \in \Lambda(x_0) \right\}.$$

Proof. See Guigues (2014a). □

3. Problem formulation

Consider the convex optimization problem

$$\begin{cases} \inf_{x_1,\ldots,x_T} \sum_{t=1}^{T} f_t(x_t, x_{t-1}) \\ x_t \in \mathcal{X}_t, \; g_t(x_t, x_{t-1}) \le 0, \quad A_t x_t + B_t x_{t-1} = b_t, t = 1, \ldots, T, \end{cases} \tag{3.1}$$

for x_0 given and the corresponding dynamic programing equations

$$\mathcal{Q}_t(x_{t-1}) = \begin{cases} \inf_{x_t} F_t(x_t, x_{t-1}) := f_t(x_t, x_{t-1}) + \mathcal{Q}_{t+1}(x_t) \\ x_t \in \mathcal{X}_t, \; g_t(x_t, x_{t-1}) \le 0, \quad A_t x_t + B_t x_{t-1} = b_t, \end{cases} \tag{3.2}$$

for $t = 1, \ldots, T$, with $\mathcal{Q}_{T+1} \equiv 0$, and $g_t(x_t, x_{t-1}) = (g_{t,1}(x_t, x_{t-1}), \ldots, g_{t,p}(x_t, x_{t-1}))$ with $g_{t,i} : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$.

Cost-to-go function $\mathcal{Q}_{t+1}(x_t)$ represents the optimal (minimal) total cost for time steps $t + 1, \ldots, T$, starting from state x_t at the beginning of step $t + 1$.

We make the following assumptions (H1):

(H1) Setting $\mathcal{X}_t^\varepsilon := \mathcal{X}_t + \varepsilon\mathbb{B}_n$, for $t = 1, \ldots, T$,

(a) $\mathcal{X}_t \subset \mathbb{R}^n$ is nonempty, convex, and compact.
(b) f_t is proper, convex, and lower semicontinuous.
(c) setting $g_t(x_t, x_{t-1}) = (g_{t,1}(x_t, x_{t-1}), \ldots, g_{t,p}(x_t, x_{t-1}))$, for $i = 1, \ldots, p$, the ith component function $g_{t,i}(x_t, x_{t-1})$ is a convex lower semicontinuous function.
(d) There exists $\varepsilon > 0$ such that $\mathcal{X}_t^\varepsilon\times\mathcal{X}_{t-1} \subset \text{dom}(f_t)$ and for every $x_{t-1} \in \mathcal{X}_{t-1}^\varepsilon$, there exists $x_t \in \mathcal{X}_t$ such that $g_t(x_t, x_{t-1}) \le 0$ and $A_t x_t + B_t x_{t-1} = b_t$.
(e) If $t \ge 2$, there exists

$$\bar{x}_t = (\bar{x}_{t,t}, \bar{x}_{t,t-1}) \in \text{ri}(\mathcal{X}_t)\times\mathcal{X}_{t-1} \cap \text{ri}(\{g_t \le 0\})$$

such that $\bar{x}_{t,t} \in \mathcal{X}_t$, $g_t(\bar{x}_{t,t}, \bar{x}_{t,t-1}) \le 0$ and $A_t\bar{x}_{t,t} + B_t\bar{x}_{t,t-1} = b_t$.

Comments on the assumptions. Assumptions (H1)-(a)–(H1)-(c) ensure that the cost-to-go functions $\mathcal{Q}_t, t = 2, \ldots, T$ are convex.

Assumption (H1)-(d) guarantees that $\mathcal{Q}_t$ is finite on $\mathcal{X}_{t-1}^\varepsilon$ and has bounded subgradients on $\mathcal{X}_{t-1}$. It also ensures that the cut coefficients are finite and therefore that the lower piecewise affine approximations computed for $\mathcal{Q}_t$ by the DDP algorithm are convex