European Journal of Operational Research 000 (2016) 1–12

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Discrete Optimization

Fixed charge multicommodity network design using *p*-partition facets

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ARTICLE INFO

Article history:
Received 26 September 2015
Accepted 7 September 2016
Available online xxx

Keywords:
Fixed charge
Multicommodity flows
Network design
Integer programming
Polyhedral structure

ABSTRACT

We are given an undirected network $G[V, E]$ and a set of traffic demands. To install a potential edge $e \in E$ we incur a cost F_e to provide a positive capacity a_e. The objective is to select edges, at minimum cost, so as to permit a feasible multicommodity flow of all traffic. We study structure of the projection polytope of this problem, in the space of binary variables associated with fixed-charges, by relating facets of a p node problem ($p = 2, 3,$ or 4), defined over a multi-graph obtained by a p-partition of V, to the facets of the original problem. Inspired from the well-known "cover" inequalities of the Knapsack Problem, we develop the notion of p-partition cover inequalities. We present necessary and sufficient conditions for such inequalities to be facet defining for $p = 3$ and 4. A simple heuristic approach for separating and adding such violated inequalities is presented, and implemented for p values up to 10. We report optimal solutions for problems with 30 nodes, 60 edges, and fully dense demand matrices within a few minutes of cpu time for most instances. Some results for dense graph problems are also reported.

1. Introduction

A general problem that arises in network design is to install capacities on a subset of edges, from a given set of available edges, and to route the demands for different commodities (source–destination pairs) over these installed edges subject to edge capacities and other design constraints, with the objective of minimizing total capacity installation and routing costs. The model has wide applications in telecommunication and transportation planning.

Research on the Network Design Problem can be broadly divided into two categories, depending upon how these capacities are installed: The (Facilities) Network Design Problem (FND) and the Fixed Charge Network Design Problem (FCND).

The FND involves one or more facility types, with different unit capacities, and we are allowed to install on an edge, any (integer) number of each of these facility types. Thus the decision variables associated with installing capacities are not binary but general integer variables. This model was first introduced in Magnanti, Mirchandani, and Vachani (1993) where the problem was defined over an undirected network $G[V, E]$ with $n = |V|$ nodes and $m = |E|$ edges, with only one facility type. The model incorporates only capacity installation costs, and no routing costs, with the following mixed-integer programming formulation:

$$\min \sum_{\{i,j\}\in E} F_{ij}x_{ij} \tag{1}$$

$$\sum_{j:\{i,j\}\in E} y^k_{ij} - \sum_{j:\{i,j\}\in E} y^k_{ji} = \begin{cases} d^k, & \text{if } i = O(k) \\ -d^k, & \text{if } i = D(k) \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

$$\sum_{k\in K}(y^k_{ij} + y^k_{ji}) \le x_{ij}, \quad \forall\{i, j\} \in E \tag{3}$$

$$y^k_{ij}, y^k_{ji} \ge 0, \quad \forall\{i, j\} \in E, \quad \forall k \in K$$

$$x_{ij} \ge 0, \quad \text{integer}, \quad \forall\{i, j\} \in E$$

In this formulation, E is the set of (undirected) edges, and commodity k has origin $O(k)$, and destination $D(k)$. Assume, without loss of generality, that each unit of the facility has a capacity of 1. The continuous variable y^k_{ij} models the flow of commodity k on edge $\{i, j\}$ in the direction i to j, and the variable x_{ij} models the integer number of units of the facility installed on edge $\{i, j\}$. Constraint in (2) are the flow-conservation equations. The edge-capacity constraints in (3) limit the total amount of flow of all commodities on each edge in both direction.

Several polyhedral results on this problem (Agarwal, 2006; Avella, Mattia, & Sassono, 2007; Bienstock, Chopra, Gunluk, & Tsai, 1998) focus on the capacity formulation that is obtained by projecting out all flow-variables. Article (Agarwal, 2006) introduces

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http://dx.doi.org/10.1016/j.ejor.2016.09.015

Please cite this article as: Y.K. Agarwal, Y.P. Aneja, Fixed charge multicommodity network design using *p*-partition facets, European Journal of Operational Research (2016), http://dx.doi.org/10.1016/j.ejor.2016.09.015

the notion of a p-partition of a network, which results in a p-node problem, and shows that under a mild condition, the facets of the p-node problem correspond to the facets of the original problem. This model has clear applications in telecommunication networks where routing costs are negligible compared to capacity installation costs. Most of the studies of this model use undirected networks.

In the FCND, to send flow through an arc we have to install a fixed capacity for a fixed cost on that arc. So the decision variables for capacity installation are binary variables. This general model was studied in Magnanti and Wong (1984) illustrating many applications in the fields of logistics and transportation. Given a directed network $G[V, A]$, where V is the set of nodes, and A is the set of arcs, and a set of commodities, with origin–destination pairs $\{(O(k), D(k), k \in K\}$ with demands $\{d^k, k \in K\}$ to be routed from origins to their respective destinations, the objective is to minimize the sum total of routing costs and capacity installation costs. The model was formulated as the following mixed integer program:

$$\min \sum_{k\in K} \sum_{(i,j)\in A} f_{ij}^k y_{ij}^k + \sum_{(i,j)\in A} F_{ij} x_{ij} \tag{4}$$

$$\sum_{j:(i,j)\in A} y_{ij}^k - \sum_{j:(j,i)\in A} y_{ji}^k = \begin{cases} d^k, & \text{if } i = O(k) \\ -d^k, & \text{if } i = D(k) \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

$$\sum_{k\in K} y_{ij}^k \le a_{ij} x_{ij}, \quad \forall (i, j) \in A \tag{6}$$

$$y_{ij}^k \ge 0, \forall (i, j) \in A, \quad \forall k \in K \tag{7}$$

$$x_{ij} \in \{0, 1\}, \quad \forall (i, j) \in A \tag{8}$$

Constraints in (6) ensure that any arc (i, j) is permitted to route the flow, up to a maximum of a_{ij}, only if $x_{ij} = 1$, resulting in a fixed cost of F_{ij}. The per-unit cost of shipping one unit of commodity k on arc (i, j) is represented by f_{ij}^k.

Most of the papers on the FCND deal with a directed network and incorporate both variable (flow) costs as well the fixed costs in the objective function. Since LP-relaxation generally provides a weak lower bound (Crainic, Frangioni, & Gendron, 1999), Lagrangian relaxation procedures (Crainic et al., 1999; 2001) have been used to improve the lower bound. In Chouman, Crainic, and Gendron (2011), authors present a cutting-plane algorithm using several classes of valid inequalities.

To the best of our knowledge, there are no polyhedral results in the literature on the FCND.

In this paper, we consider the problem over an undirected network (UFCNDP-F) with only fixed costs, as considered in Magnanti et al. (1993) with the following MIP formulation:

$$\min \sum_{\{i,j\}\in E} F_{ij} x_{ij} \tag{UFCNDP-F}$$

$$\sum_{j:\{i,j\}\in E} y_{ij}^k - \sum_{j:\{i,j\}\in E} y_{ji}^k = \begin{cases} d^k, & \text{if } i = O(k) \\ -d^k, & \text{if } i = D(k) \\ 0, & \text{otherwise,} \end{cases} \tag{9}$$

$$\sum_{k\in K} (y_{ij}^k + y_{ji}^k) \le a_{ij} x_{ij}, \quad \forall \{i, j\} \in E \tag{10}$$

$$y_{ij}^k, y_{ji}^k \ge 0, \quad \forall \{i, j\} \in E, \quad \forall k \in K \tag{11}$$

$$x_{ij} \in \{0, 1\}, \quad \forall \{i, j\} \in E \tag{12}$$

This model differs from the one in Magnanti et al. (1993) since the capacity installation variables in (12) are binary, and constraint set (10) ensures that the capacity of an edge $\{i, j\}$ is either a_{ij} or 0, depending upon whether capacity is installed (with cost F_{ij}) or not. We will assume, without loss of generality, that $a_e > 0$ for all edges $e \in E$. This model has been studied in Zaleta and Socarras (2004), Lewis (2009), Herrmann, Ioannou, and Proth (1996) and Balakrishnan, Magnanti, and Wong (1989). All these papers attempt to find good solutions to this problem. While authors in Zaleta and Socarras (2004) use Tabu Search metaheuristic to find good quality solutions, article (Lewis, 2009) presents a preprocessing technique for identifying critical edges, using design of experiment (DOE) principles. Authors in Herrmann et al. (1996) present a dual-ascent procedure for finding “near optimal” solutions to the problem by extending the dual-ascent approach proposed by Balakrishnan et al. (1989) for the uncapacitated version of the problem. Gendron in Gendron (2002) showed that the approach proposed in Herrmann et al. (1996) was incorrect, and suggested a simple modification to fix the error. It was also argued in Gendron (2002) that corrected dual-ascent approach was not very effective for the problem.

The outline of the article is as follows. In Section 2 we present the projection polytope obtained by projecting out the flow variables. In Section 3 we consider a p-node multi-graph by considering a p-partition of the node set V and shrinking nodes in each subset of the partition into a single node. We then present a theorem which relates facet defining inequalities of this projection polytope to facet defining inequalities of corresponding polytope of the “aggregate” problem defined over this p-node multi-graph. In Section 4, we study the 3-node multigraph and develop necessary and sufficient conditions for “cover inequalities”, that we introduce, to be facet-defining for convex hull of feasible solutions of this 3-node problem. Section 5 extends the study to a 4-node problem. Section 6 explains how the idea for cover inequalities can be extended to larger values of p, and Section 7 described p-partition based spanning tree inequalities. Section 8 discusses the flow versus the capacity formulation of the problem, and demonstrates the computational superiority of the latter with a new implementation. In Section 9 we describe the implementation details, and the separation heuristics used to obtain the violated cover inequalities. Finally in Section 10, we present our computational experiments which demonstrate the effectiveness of our approach. Some concluding remarks are made in Section 11.

2. The projection polytope

According to a well-known theorem on multi-commodity flows in Iri (1971) and in Onaga and Kakusho (1971), the flow conservation constraints in formulation UFCNDP-F can be replaced by constraints of the form $\mu(a-d) \ge 0$, where a and d are vectors of capacities and demands, respectively, defined on the node-pairs of $G[N, E]$, and μ is a metric. A non-negative symmetric function μ defined on all pairs of nodes of G, is said to be a metric if for any three distinct nodes $x, y, z \in N$, $\mu(\{x, y\}) \le \mu(\{x, z\}) + \mu(\{z, y\})$. For $\{i, j\} \notin E$, we set $a_{ij} = 0$, and for commodities $\{i, j\} \notin K$, we set $d_{ij} = 0$. The theorem, popularly referred to as the *Japanese Theorem*, asserts that for any multi-commodity flow problem, a capacity vector a is feasible for a demand vector d if and only if $\mu(a-d) \ge 0$ for every metric μ.

The set of all metrics is infinitely large and forms a polyhedral convex cone. Any extreme ray of this cone is called a *primitive* metric. Since any metric can be expressed as a convex combination of *primitive* metrics, for feasibility of a multi-commodity flow it is enough that the condition $\mu(a-d) \ge 0$ holds only for the set $\mathcal{M}$ of all *primitive* metrics. This provides an IP formulation for the (UFCND) that involves only the binary capacity variables:

$$\min \sum_{e\in E} F_e x_e \tag{UFCND}$$