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Sharing and growth in general random multiplicative environments

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ABSTRACT

We extend the discrete-time cooperation evolution model proposed by Yaari and Solomon (2010) to a version with independent multiplicative random increments of arbitrary distribution and we develop a model in continuous time driven by exponential Lévy processes. In all settings, we prove that members of a sharing group enjoy a higher growth rate of their wealth and at the same time drastically reduce their exposure to random fluctuations: as more members join the group, the wealth of each member uniformly converges to a deterministic process growing at the highest possible rate. Thus, joining a sufficiently large sharing group may promise all its members to succeed almost surely even in environments where non-sharing individuals cannot escape misery on their own.

1. Introduction

Joining a group of sharing individuals might allow its members to survive even in environments in which all non-sharing individuals would perish. We will investigate the mechanism behind successful sharing in a broad variety of model environments. As a starting point, we choose the model of Yaari and Solomon (2010). They study a dynamic discrete-time model of the evolution of an economy or environment where in every time-step individuals randomly lose or gain some fixed proportion of their wealth. Thus, nature randomly changes the individuals' wealth in a multiplicative way in every time-step. The random multiplicative increments are assumed to be independent and identically distributed. If the mean of the multiplicative increments is greater than one, every individual's expected wealth increases. Nevertheless, if the mean of the logarithm of these increments is less than zero, with probability one the individual will face a future where its wealth trajectory converges to zero. A remedy to escape this almost sure convergence to zero is cooperation: After each time-step, the population pools its wealth and evenly redistributes this pool before going into the next round. Assuming that each individual in a population faces random increments that are independent of the others' increments, Yaari and Solomon (2010) show that all individuals may be able to achieve wealth trajectories which almost surely diverge to infinity if they share their wealth with sufficiently many others. Thus, in the multiplicative model environment, cooperation through sharing arises because each individual benefits from such prima facie altruistic behaviour.

Are the benefits of sharing robust not just across model parameters but across different classes of distributions and models? To answer this question, we first extend the model of Yaari and Solomon (2010) to multiplicative increments of arbitrary distribution. Then, we develop a continuous-time model of sharing based on Lévy processes. Investigating such a broad variety of models will show whether individuals can be sure that joining a sharing group is beneficial even under uncertainty regarding the random environment they face.

Indeed, as more members join the group, growth rates are strictly increasing. Qualitative differences between environments nevertheless exist: In a broad variety of models, including all models with increments of a finite size, the maximal growth rate attainable by large groups is limited. Other models in discrete and continuous time exist where – even if all individuals would fail on their own – the maximal growth rate attainable by large groups is unlimited.

While growth rates determine the fate in the distant future, the behaviour of the wealth trajectory of individual group members on finite time intervals is investigated via uniform convergence assuring that, as more members join the group, not just single points in time but the whole trajectory approaches a deterministic behaviour with maximal growth rate.

Economic literature points out various benefits of the redistribution of income or wealth, such as social stability discussed in Forbes (2000), avoidance of high degrees of inequality as in Fehr and Schmidt (1999) and Bolton and Ockenfels (2000), or increased economic growth due to more efficient investment and better endowment of the poor as in Aghion and Bolton (1997).

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Multiplicative stochastic processes are also used to model income and wealth by e.g. Grochulski and Piskorski (2010).

Simulating risky human capital as a multiplicative stochastic process in discrete time with normally distributed logarithmic increments, Lorenz, Paetzel and Schweitzer (2013) compare various taxing schemes and show that the redistribution of wealth organised through taxes can improve economic growth through a variant of the portfolio rebalancing effect that dates back to Kelly (1956) and also allows the success of e.g. the sharing strategies investigated by Yaari and Solomon (2010) or in the present article. Simulation allows to investigate almost arbitrarily complex redistribution schemes and the resulting growth rate for a large but finite number of individuals, while only some extreme cases such as no or total taxation can be investigated analytically. Bouchaud (2015) directly models the individuals' growth rates as a stochastic process and passes to the limiting rate for an infinite number of individuals to obtain a model that allows an analytical investigation of optimal tax rates.

While the aforementioned literature focuses on economic phenomena in human societies, our discussion and more generally the setting of multiplicative random increments is not limited to this context. Peters and Adamou (2015), who introduce a model of sharing in an evolutionary context via a discrete-time approximation in a geometric Brownian motion model, mention as examples of sharing in a multiplicative random environment the cooperation of cells, animals, or humans sharing food or material resources in an organism, pack, family, company or state. We would like to add that humans may also cooperate to obtain immaterial resources such as knowledge or technology which are more quickly obtained from a higher level of prior abilities such that a multiplicative model may be more realistic than an additive one for immaterial achievements as well. While we shall see that also smaller entities such as families can benefit from sharing, societies cooperating on a larger and possibly even global scale can diminish idiosyncratic randomness and enjoy the uniform convergence leading to more reliable growth.

Aspects of cooperation different to those considered in the present article or the articles mentioned before include cooperative services, game theory, and logistics; readers interested in these additional topics may consult e.g. Chakravarthy (2016), Guajardo and Rönnqvist (2015), Guajardo and Rönnqvist (2016), Kimms and Kozeletskyi (2016), Nguyen and Thomas (2016) and the references therein.

The next section introduces the multiplicative random environment in discrete time and discusses the fate of non-sharing individuals in this environment, whereas Section 3 investigates the benefits of sharing. Examples and simulations in Section 4 illustrate these theoretical results. Section 5 introduces and investigates our continuous-time model, shows the benefits from sharing in this model in the long run through an improved growth rate and also on finite time intervals through uniform convergence of trajectories, and derives a lower bound for the minimal group size needed to assure growth. Section 6 presents examples of model types within our continuous-time setting. Furthermore, we consider a situation where individuals share only part of their wealth in Section 7.

2. Non-sharing individuals

Let $Y_t = \exp\{X_t\}$ denote an individual's wealth. Depending on the application of the model, 'wealth' should be understood as any quantity representing well-being, resources or achievements of an individual person, animal, or enterprise in any kind of risky effort, not necessarily only financial or material resources. The main question to be answered by the modeller is whether the assumed structure of multiplicative increments is sufficiently close to reality.

We assume the process to start at one, i.e. $Y_0 = 1$. The concrete initial value is irrelevant because our considerations are not affected by the choice of a specific unit measuring initial wealth.

In this section, we consider the discrete time setting and denote by $\Delta X_t = (X_t - X_{t-1})$ the independent and identically distributed increments of logarithmic wealth, such that

$$Y_t = \prod_{k=1}^{t} e^{\Delta X_k}.$$

The logarithm μ of the expectation of a multiplicative increment and the expectation r of a logarithmic increment are

$$\mu := \log \mathbb{E}[Y_1] = \log \mathbb{E}[\exp\{X_1\}]$$
$$r := \mathbb{E}[\log(Y_1)] = \mathbb{E}[X_1].$$

Jensen's inequality shows that $r \le \mu$ with a strict inequality $r < \mu$ unless increments are deterministic. In the present discrete-time setting $\log \mathbb{E}[\exp\{\Delta X_t\}] = \mu$ and $\mathbb{E}[\Delta X_t] = r$ for all t because increments in each unit of time are identically distributed.

Asymptotically, the expected wealth behaves as

$$\mathbb{E}[Y_t] = (\mathbb{E}[Y_1])^t = e^{\mu t} \overset{(t\to\infty)}{\longrightarrow} \begin{cases} \infty, & \mu > 0 \\ 1, & \mu = 0 \\ 0, & \mu < 0 \end{cases}$$

while if X is not constant, Proposition 10 in the Appendix assures that the individual's wealth trajectory behaves almost surely as

$$Y_t = e^{X_t} \overset{(t\to\infty)}{\longrightarrow} \begin{cases} \infty, & r > 0 \\ \text{oscillation}, & r = 0 \\ 0, & r < 0. \end{cases}$$

For $r < \mu < 0$, each individual's wealth vanishes in expectation and almost surely. Conversely, for $0 < r < \mu$, each individual's wealth converges to infinity in expectation and almost surely. If $r < 0 < \mu$, however, the expected wealth grows to infinity while the wealth trajectory converges to zero almost surely. In this situation, individuals on their own are lost despite growing expected wealth. We are going to investigate how individuals can change their fate for the better by gathering in groups and sharing their resources.

3. Sharing in discrete time

3.1. Synchronous sharing in discrete time

Let us now consider a group of N sharing individuals. Assume that after each time step, they pool their wealth and equally distribute the wealth pool among them. Thus, after sharing at time $t-1$, each individual holds one of the N equal shares denoted by $Y_{t-1}^{(N)}$ in the group's pooled resources $NY_{t-1}^{(N)}$. Then, each individual l faces an individual random logarithmic increment $\Delta X_t^{\langle l\rangle}$ that is independent of the random events faced by all other members of the group and returns with $Y_{t-1}^{(N)} e^{\Delta X_t^{\langle l\rangle}}$.

Now, when the new pool is shared at time t, each individual receives an equal share

$$Y_t^{(N)} = \frac{1}{N}\sum_{l=1}^{N} Y_{t-1}^{(N)} e^{\Delta X_t^{\langle l\rangle}} = Y_{t-1}^{(N)} \frac{1}{N}\sum_{l=1}^{N} e^{\Delta X_t^{\langle l\rangle}}.$$

The logarithmic share $X_t^{(N)} := \log(Y_t^{(N)})$ is the sum of its increments

$$\Delta X_t^{(N)} = \log\left(\frac{Y_t^{(N)}}{Y_{t-1}^{(N)}}\right) = \log\left(\frac{1}{N}\sum_{l=1}^{N} e^{\Delta X_t^{\langle l\rangle}}\right).$$

These increments are independent and identically distributed because all individual logarithmic increments $\Delta X_t^{\langle l\rangle}$ are independent and identically distributed.