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Interfaces with Other Disciplines

Optimal investment and consumption when allowing terminal debt

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ABSTRACT

We analyze a dynamic optimization problem which involves the consumption and investment of an investor with constant relative risk aversion for consumption but with a risk aversion for final wealth which does not necessarily imply that terminal wealth must always be positive. We require risk aversion for terminal wealth to be positive but not monotone: there is a point of maximal risk aversion at zero wealth and the investor may continue to consume when wealth is negative. Using dual optimization methods we can derive explicit solutions and we find that the optimal solution differs in a fundamental way from the case where risk aversion is monotone. It turns out that the optimal consumption function is convex and concave at different wealth levels and that the optimal investment strategy may no longer be monotone as a function of the remaining time to invest and consume.

1. Introduction

The present paper contributes to the existing literature on optimal portfolio theory by finding explicit solutions for an investor who is risk-averse but who is willing to end up with debts, i.e. a negative wealth at the end of the consumption and investment period. Closed-form optimal policies have, to the best of our knowledge, only been found for the specific class of exponential utilities, where risk aversion is independent of wealth. Instead, we formulate a model to study the consumption patterns for people who try to avoid debt, but do not avoid it at all costs. In this model, there is therefore an explicit tradeoff between a higher probability of debts at the end of the time period under consideration and the utility of immediate consumption.

A specific example which can motivate our approach would be a fund manager who invests to generate cashflows for a client during a finite time period. She would try to avoid negative wealth at the end of the investment period since her performance measurement over the investment period will take both consumption during and wealth after the period into account, but she may not exclude negative wealth on beforehand to allow the possibility to 'gamble for resurrection' in disadvantageous scenarios.

Such behavior could be interpreted as a 'sunk costs effect' (also known as 'escalation of commitment'), which in this case would mean that the investor, having been unsuccessful enough on the stock market to arrive at negative wealth levels, would feel an incentive to continue investing in stocks to make up for the lost earlier investments.[2]

In that sense our model is meant to be descriptive rather than normative and it assumes that the investor still has the possibility to borrow money once wealth is negative. This may be the case, for example, when the wealth position is not disclosed during but only after the investment period.[3] Our setup also allows us to deal with cases where an investor starts with negative wealth at the initial point in time but still shows (asymptotically) constant relative risk aversion for high positive wealth levels.

The standard investment and consumption model which assumes preferences with constant relative risk aversion (CRRA) cannot be applied for such investors since it assumes that wealth must stay positive and preferences with constant absolute risk aversion (CARA) allow only risk aversion which does not depend on wealth. We thus believe it may be useful to offer an alternative which takes account of such possibilities. In particular, we would like to

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[2] See Arkes and Blumer (1985) for the psychology of sunk costs, Thaler and Johnson (1990) for the break-even effect and the paper by Baghestanian and Massenot (2015) for recent empirical evidence on the incentive to gamble for resurrection.

[3] If wealth levels would be disclosed during the investment period, it would be more natural that the interest rate on the cash account would change once wealth becomes negative. We will not make that assumption here; see He and Pagés (1993) and Dybvig and Liu (2010) for results when borrowing is restricted and Korn (1995) for the case where there is a higher interest rate for borrowing than for lending. It seems that in the latter case, our optimization problem can only be solved numerically.

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allow investors to have different preferences for consumption and for terminal wealth, but we still want explicit formulas for the optimal strategies that can be directly compared to strategies in which terminal debts must be avoided at all costs or where investors become risk-seeking when wealth becomes negative.

1.1. SAHARA utility

We therefore assume that an investor has CRRA preferences for consumption and a utility for terminal wealth which exhibits Symmetric Asymptotic Hyperbolic Absolute Risk Aversion (SAHARA). This class of utility functions allows for positive and negative values of wealth. For very large values the risk preferences correspond to those of the CRRA class but SAHARA investors have a level of wealth at which their risk aversion is maximal.[4] We take this reference point, without loss of generality,[5] to be the point of zero wealth and we may interpret the fact that the investor becomes more risk averse when the point of zero wealth is approached from above as a manifestation of zero-risk-bias. However, once an investor has crossed this point and wealth has become negative, she becomes less risk averse if wealth becomes even more negative. This property formalizes the intuition that people may become less concerned about debt once they are already in debt. If that would not be the case, there would, for example, be no people who switch to a new credit card once their first card has a negative balance which is not compensated by other assets they own.

The SAHARA utility functions define a less risk averse attitude for large values of wealth but also for large values of debt, i.e. for very negative wealth. However, we will still assume that there is positive risk aversion once the investor is heavily in debt. We do not include risk seeking behavior such as proposed in models with gain-loss preferences (see Tversky & Kahneman (1992)). We thus retain concavity of utility as a function of terminal wealth on the entire domain, which allows us to find explicit solutions for the optimized behavior in this dynamic setting. We will compare our optimized strategies to those based on gain-loss preferences in the last section before the conclusion.

The optimal consumption and asset allocation problem in a continuous-time setting dates back to Merton (1969, 1971). The solutions to the associated stochastic optimal control problems strongly depend on risk preferences and the assumptions on the asset price dynamics. The most frequently used utility functions exhibit hyperbolic absolute risk aversion (HARA). In particular, one often assumes that either wealth has to be above a certain threshold (for example, by assuming CRRA utility functions) or that individuals' risk aversion does not depend on their wealth (for CARA utility functions). Together with the assumption of Brownian or Geometric Brownian asset dynamics, Merton shows that closed-form solutions can be obtained for these risk preferences.[6] In Chen, Pelsser, and Vellekoop (2011) this closed-form solution for Merton's problem is extended to include SAHARA risk preferences. This class of utility functions also arises naturally in the framework of forward performance criteria: see the earlier formulations in Zariphopoulou and Musiela (2009, 2010).

In contrast to the analysis in Chen et al. (2011) we consider both the utility of intermediate consumption and the utility of terminal wealth in our optimization problem. Since it is possible to invest in riskless and risky assets during this period, we thus allow the investor to postpone the payments of debts to a later time while continuing to consume, in the hope that favorable returns on financial assets may still help to achieve a better level of wealth at the end.

1.2. Dual methods

Since we will not make the Markovian assumption for the asset price dynamics, we rely on the dual approach for optimization in stochastic dynamic systems to derive explicit solutions in the general case. Such methods have been studied for complete markets in Cox and Huang (1989), Karatzas, Lehoczky, and Shreve (1987) and Pliska (1986) and in an incomplete market setting e.g. by Davis (1997), He and Pearson (1991a,b), Karatzas, Lehoczky, Shreve, and Xu (1991), Schachermayer (2001) and Rogers (2003). We apply the dual method in a multi-asset economy which takes survival probabilities of the investor into account. If Markovian dynamics are not assumed we cannot necessarily express the optimal consumption strategy as an explicit function of the level of wealth. But if we assume that the parameters which specify the market dynamics and preferences are deterministic functions of time, closed-form strategies can be derived which provide direct insight in consumption patterns and the optimal portfolio holdings.

We find for example that if asymptotic risk aversion for large values of terminal wealth is larger (or smaller) than the risk aversion for consumption, the wealth level at which investment in risky assets is minimal occurs for a negative (or positive) value. If these two risk aversion coefficients are equal, we can show how much the level of consumption and the allocation to risky assets increase in comparison to an investor who has CRRA preferences for *both* consumption and terminal wealth. This difference is time-varying and involves several key parameters, such as the strength of the propensity to become less risk averse when going further into debt (measured by a parameter β), the tradeoff between utility from consumption and terminal wealth (measured by a parameter K) and a parameter ν_γ which is a combination of the risk free rate r, asymptotic risk aversion γ, the vector of market prices of risk θ, the discount factor δ and force of mortality ς (see Corollary 3.6).

We also find that over long time horizons the investment strategy does not differ that much from the one generated by a loss averse preference, which is one of the characteristic elements in cumulative prospect theory (see Berkelaar, Kouwenberg, and Post (2004) and Jin and Zhou (2008)). One can interpret our choice as in between the classical framework of CRRA preferences, where negative wealth is avoided at all cost, and gain-loss preferences, which are risk seeking for negative wealth. In our case, the investor remains risk averse for negative wealth but less so if wealth becomes more negative, so in that case the investor gets closer to the point where risk aversion becomes risk seeking behavior, without ever reaching it.

The remainder of the paper is structured as follows. Section 2 describes the underlying financial market and formulates

[4] SAHARA utility assumes a unique reference point at which the absolute risk aversion for terminal wealth reaches its maximum. In this sense, SAHARA utility does not capture the preferences of investors who exhibit *multiple* reference points where risk aversion levels obtain a *local* maximum. Some recent experimental studies have shown that investors may indeed display multiple reference points in terminal wealth, see Knoller (2016), and Koop and Johnson (2012). Wang and Johnson (2012) develop a theory based on three reference points: the minimum requirement value, the status quo, and the goal. We believe that our study of preferences with a single point of maximal risk aversion provides insight into the more complicated case where there are more reference points.

[5] Choosing the reference point *d* differently for the utility of terminal wealth (and for the levels of consumption) would amount to a translation of the whole optimization problem and the corresponding optimal strategies over the wealth axis, since our asset returns do not depend on wealth. Note, however, that this is no longer true if there would be multiple local maxima for risk aversion.

[6] Merton's optimal consumption and investment problem has been further investigated and extended in different contexts. Some relevant references are Dybvig and Huang (1988), Davis and Norman (1990), Sørensen (1999), Brennan and Xia (2000), Munk and Sørensen (2004, 2010), Korn and Steffensen (2007), Dybvig and Liu (2010), and Kraft and Munk (2011). For the case of investors which are not assumed to be time-consistent see Marín-Solano and Navas (2010).

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