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Discrete Optimization

An extension of the Christofides heuristic for the generalized multiple depot multiple traveling salesmen problem

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ABSTRACT

We study a generalization of the classical traveling salesman problem, where multiple salesmen are positioned at different depots, of which only a limited number (k) can be selected to service customers. For this problem, only two 2-approximation algorithms are available in the literature. Here, we improve on these algorithms by showing that a non-trivial extension of the well-known Christofides heuristic has a tight approximation ratio of $2 - 1/(2k)$. In doing so, we develop a body of analysis which can be used to build new approximation algorithms for other vehicle routing problems.

1. Introduction

We study a generalization of the classical traveling salesman problem (TSP), commonly referred to as the *Generalized Multiple Depot Multiple Traveling Salesmen Problem*, or GMDMTSP for short (Malik, Rathinam, & Darbha, 2007). Given a set of multiple salesmen positioned at different depots, the objective of the GMDMTSP is to select at most $k \geq 1$ salesmen to service customers situated at different cities through closed walks (or cycles) so as to minimize the total travel distance. This problem has a wide range of applications, and can be found, for example, in routing unmanned aerial vehicles (Chandler & Pachter, 1998; Chandler et al., 2002), as well as in location and routing optimization for air ambulance services (Carnes & Shmoys, 2011; Prodhon & Prins, 2014).

The GMDMTSP can be defined on a complete undirected graph $G = (V, E)$ with a vertex set V and an edge set E. Let n indicate the number of vertices. Let $D \subseteq V$ denote a set of depots where each depot represents the base location of a distinct salesman. Let $I := V \backslash D$ denote a set of customers where each customer in I is located in a city. Each edge $(u, v) \in E$ has a non-negative length $\ell(u, v)$, indicating the distance of the locations of u and v. We assume that the edge lengths are symmetric and satisfy the triangle inequality. Let k indicate the maximum number of salesmen that can be selected to visit customers, where $1 \leq k \leq |D|$. A feasible solution is thus a collection of at most k cycles that include each customer exactly once, and where each cycle begins and ends at a distinct depot. The objective in the GMDMTSP is to find a feasible solution that minimizes the total cycle length.

As with the TSP, the GMDMTSP is strongly NP-hard. Hence, it is of practical interest to develop constant ratio approximation algorithms. Here, we recall that, for a minimization problem, an algorithm is a ρ-approximation algorithm with an approximation ratio ρ if it has a polynomial running time and always provides a solution with a value no more than ρ times the minimum objective value. The ratio ρ is tight if there exists an instance for which the solution obtained has a value equal to ρ times the minimum objective value.

For the GMDMTSP, only two 2-approximation algorithms are available in the literature (Carnes & Shmoys, 2011; Malik et al., 2007). Both have polynomial time complexities, with one being nearly $O(n^4)$, and the other being $O(n^2 \log n)$. Here, we improve on these algorithms by providing a new non-trivial extension of the well-known Christofides heuristic of the TSP (Christofides, 1976) which has a tight approximation ratio of $2 - 1/(2k)$ and time complexity nearly $O(n^4)$.

The rest of the paper is organized as follows: following a literature review in Section 2 and preliminaries in Section 3, we develop an extension of the Christofides heuristic for the GMDMTSP in Section 4, and go on to prove that it has a tight approximation ratio of $2 - 1/(2k)$ in Section 5. The proof requires an inequality, which is shown in Section 6. We conclude this paper in Section 7.

2. Literature review

For the GMDMTSP, only two 2-approximation algorithms are known, and they both extend a tree algorithm of the TSP. This

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algorithm for the TSP has three steps (Papadimitriou & Steiglitz, 1998; Rosenkrantz, Stearns, & Lewis, 1977): (1) find a minimum spanning tree (MST) of the given graph; (2) construct an Eulerian multigraph by duplicating all edges of the MST; and (3) find an Eulerian closed walk of the multigraph, remove repeated vertices of the closed walk, and return the resulting cycle. In the TSP, since the MST can be no longer than the optimal cycle, the tree algorithm has an approximation ratio of 2. Since the MST and the Eulerian closed walk can be obtained in $O(n^2)$ time and $O(n)$ time, respectively, the tree algorithm for the TSP runs in $O(n^2)$ time.

Malik et al. (2007) extended the tree algorithm for the GMDMTSP by introducing a *degree constrained spanning forest (DCSF) w.r.t.* (G, D, k), which is defined as a spanning forest of G that covers all the vertices in V, with each tree of the forest containing a distinct depot in D as its root, and with the total degree of all the roots not exceeding k. It is known that a shortest DCSF F^* w.r.t. (G, D, k) is a lower bound on the optimal solution of the GMDMTSP, and that F^* can be computed in $O(n^4\alpha^2\log^2\alpha)$ time by a Lagrangian relaxation method (Malik et al., 2007). Here $\alpha := \alpha(n^2, n)$ is the functional inverse of Ackermann's function, which grows very slowly and can be considered as a constant (Chazelle, 2000). Thus, by replacing the MST with F^*, the tree algorithm can be extended to return a feasible solution of at most twice the length of the optimal solution to the GMDMTSP, in $O(n^4\alpha^2\log^2\alpha)$ time, which is nearly $O(n^4)$ time.

Carnes and Shmoys (2011) developed another 2-approximation algorithm for the GMDMTSP, which they studied as a variant of the classical location-routing problem (Goemans & Williamson, 1995; Laporte, Nobert, & Taillefer, 1988; Mina, Jayaraman, & Srivastava, 1998). Their algorithm also extended the tree algorithm, but applied a primal and dual schema to obtain a DCSF w.r.t. (G, D, k), which may not be a shortest DCSF but is a lower bound on the optimal solution of the GMDMTSP. This algorithm is equivalent to a truncated version of the well known Kruskal's minimum spanning tree algorithm, for which the best implementation requires $O(n^2\log n)$ time (Cormen, Leiserson, Rivest, & Stein, 2001).

To improve on the existing best 2-approximation for the GMDMTSP, Malik et al. (2007) suggested extending the well-known Christofides heuristic of the TSP. The Christofides heuristic of the TSP improves on the tree algorithm by revising only Step 2, where it adds to the MST only edges of a minimum-weight perfect matching for vertices of odd degree in the MST. Since the number of vertices of odd degree in the MST is even, by short-cutting the optimal TSP tour, one can obtain the union of two disjoint perfect matchings on these vertices. It follows, by the triangle inequality, that the length of the minimum-weight perfect matching obtained in Step 2 is not longer than half of the optimal TSP tour. This guarantees that the Christofides heuristic achieves a superior ratio of 3/2 for the TSP. Since there are at most n vertices of odd degree in the MST, the number of edges that connect these vertices cannot exceed $n(n-1)/2$. Thus, the minimum-weight perfect matching can be obtained in $O(n^3)$ time (Cook & Rohe, 1999; Gabow, 1990), implying that the Christofides heuristic of the TSP runs in $O(n^3)$ time. For the GMDMTSP, it is natural to extend the Christofides heuristic of the TSP by replacing the MST with the shortest DCSF F^* w.r.t. (G, D, k). However, as pointed out by Malik et al. (2007), the worst-case analysis of this extended heuristic is challenging, since it needs to bound the length of a minimum-weight perfect matching for vertices of odd degree in F^*, for which no effective approach is available in the literature. In this paper, we now develop several new approaches to bound the edges of this matching, which allows us to show that the extended Christofides heuristic achieves a tight approximation ratio of $2 - 1/(2k)$.

Extensions of the Christofides heuristic have been proved to guarantee approximation ratios that are less than 2 for some special cases of the GMDMTSP and their variants. For the multiple depot multiple TSP (MDMTSP), a special case of the GMDMTSP with $k = |D|$, Xu, Xu, and Rodrigues (2011) showed that when $k \geq 2$, the Christofides heuristic can be extended to achieve a tight approximation ratio of $2 - 1/k$ in $O(n^3)$ time, by replacing the MST with a shortest *constrained spanning forest (CSF) w.r.t.* (G, D), where a CSF w.r.t. (G, D) is defined as a spanning forest of G that covers all the vertices in V, with each tree containing a distinct depot in D. Noting that, unlike the DCSF, the total degree of the roots of a CSF may exceed k, the worst-case analysis for the extended Christofides heuristic of the MDMTSP cannot be directly applied to that of the GMDMTSP. Besides this, Rathinam and Sengupta (2010) have extended the Christofides heuristic to obtain a 3/2-approximation algorithm that runs in $O(n^3)$ time for a two-depot Hamiltonian path problem, which determines paths instead of cycles for salesmen. The analysis of the approximation ratio in their work is manageable, since it needs only to consider a two-depot case with $k = |D| = 2$. In addition, Xu and Rodrigues (2015) have developed a 3/2-approximation algorithm for the MDMTSP, but whose running time is $O(n^{3k})$, which is exponential in k. In this paper, we develop a $[2 - 1/(2k)]$-approximation algorithm for the GMDMTSP with a polynomial running time of only about $O(n^4)$.

3. Preliminaries

Recall the definitions of *walk*, *tree*, *rooted tree*, *forest*, *matching*, and *perfect matching* in Diestel (2010). A walk, $(v_1 v_2 \ldots v_t v_{t+1})$ where $t \geq 0$, is a *closed walk* if $v_1 = v_{t+1}$. A walk with no repeated vertices is a *path*. A closed walk with no repeated vertices except its start and end vertices is a *cycle*. A multigraph is an undirected graph that may contain multiple edges between a pair of vertices. A connected multigraph is *Eulerian* if the degree of each vertex is even. Every Eulerian multigraph has an *Eulerian closed walk*, that is, a closed walk containing every edge (Diestel, 2010).

Consider any two vertices u and v of a rooted tree T with the root of T denoted by r. If u is on the unique path that connects r and v in T, then u is an *ancestor* of v, and v is a *descendant* of u. If u is an ancestor of v and (u, v) is in T, then u is the *parent* of v, and v is a *child* of u. Moreover, throughout the paper, for any subgraph H of G, we use $V(H)$, $E(H)$, and $\ell(H)$ to denote the vertex set, the edge set, and the total length of edges of H. For any edge subset W with $V(W) \subseteq V(H)$, we use $H - W$ to denote a graph on $V(H)$ with an edge set equal to $E(H)\backslash W$, and we use $H + W$ to denote a graph on $V(H)$ with an edge set equal to $E(H) \cup W$. As in Section 2, $\alpha := \alpha(n^2, n)$ indicates the functional inverse of Ackermann's function, which, as pointed out, grows very slowly and can be considered as a constant (Chazelle, 2000).

In this paper, we use F^* to indicate a shortest DCSF w.r.t. (G, D, k) as defined in Section 2, and use a cycle collection $\mathcal{C}^{\text{opt}}$ to indicate an optimal solution to the GMDMTSP.

4. An Extension of the Christofides heuristic

We elaborate on our extension of the Christofides heuristic for the GMDMTSP in Algorithm 1. It first computes a shortest DCSF F^* w.r.t. (G, D, k), and obtains a vertex set $\text{Odd}(F^*)$ that contains all the vertices of odd degree in F^*. It then computes a minimum-weight perfect matching $M^*(F^*)$ in the subgraph of G induced by $\text{Odd}(F^*)$, and adds to F^* every edge of $M^*(F^*)$ (or a copy of the edge if the edge is in F^*). As a result, it obtains a new multigraph on V, in which each vertex is guaranteed to have an even degree. Thus, each connected component of the multigraph is Eulerian, and must have an Eulerian closed walk. By removing repeated vertices and redundant depots in these closed walks, Algorithm 1 obta ins and returns a collection of cycles, denoted by $\mathcal{C}(F^*)$.

Algorithm 1 (An extended Christofides heuristic for the GMDMTSP).