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Discrete Optimization

A new lift-and-project operator

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ABSTRACT

In this paper, we analyze the strength of split cuts in a lift-and-project framework. We first observe that the Lovász–Schrijver and Sherali–Adams lift-and-project operator hierarchies can be viewed as applying specific 0–1 split cuts to an appropriate extended formulation and demonstrate how to strengthen these hierarchies using additional split cuts. More precisely, we define a new operator that adds all 0–1 split cuts to the extended formulation. For 0–1 mixed-integer sets with k binary variables, this new operator is guaranteed to obtain the integer hull in $\lceil k/2 \rceil$ steps compared to k steps for the Lovász–Schrijver or the Sherali–Adams operator. We also present computational results on the stable set problem with our new operator.

1. Introduction

For a given 0–1 mixed integer set P^{IP}, defined as the intersection of a polyhedron P and $\{0,1\}^{n_1} \times \mathbb{R}^{n_2}$, a fundamental goal in integer programing is to obtain a better approximation of its convex hull than P. A set is called a strong relaxation of the set P^{IP} if it contains P^{IP} and at the same time is strictly contained in P. Starting with the pioneering work of Gomory (1963) on cutting planes, different techniques have been developed to build strong relaxations. Lift-and-project techniques such as the ones developed by Sherali and Adams (1990), Lovász and Schrijver (1991), Balas, Ceria, and Cornuéjols (1993) and Lasserre (2001), obtain strong relaxations by first formulating a set in a higher dimensional space and then projecting this set onto the space of the original variables. In this paper, we describe a new lift-and-project operator that is closely related to the Sherali–Adams and Lovász–Schrijver (without semidefiniteness) operators. Similar to their operators, our operator also produces polyhedral relaxations in the original space. Both of these operators yield a hierarchy of relaxations $H^1, H^2, \ldots, H^{n_1}$ of P^{IP} with the following property:

$$P \supseteq H^1 \supseteq H^2 \supseteq \cdots \supseteq H^{n_1} = \mathrm{conv}(P^{IP}).$$

Therefore, these operators obtain the convex hull of P^{IP} in at most n_1 steps and there are examples where n_1 steps are necessary (Cook & Dash, 2001; Laurent, 2003). See (Conforti, Cornuejols, & Zambelli, 2014; Laurent, 2003) for a review and comparison of these hierarchies. The new operator we describe in this paper is guaranteed to obtain the integer hull in $\lceil n_1/2 \rceil$ steps instead.

In an earlier paper (Bodur, Dash, & Günlük, 2015), for every 0–1 mixed-integer set P^{IP} with n_1 integer and n_2 continuous variables we showed how to construct an extended formulation of P (which we call an *extended LP formulation* of P^{IP}) with $n_1 - 1$ additional continuous variables whose 0–1 split closure is integral. It is well-known that the 0–1 split closure of a 0–1 mixed-integer set can be computed in time bounded by a polynomial function of the encoding size of P – i.e., the number of bits required to represent the inequalities defining P (and we describe this computation more precisely later). The extended LP formulation presented in Bodur et al. (2015) is only of theoretical interest, as it requires the list of all extreme points and rays of P, which could be of exponential size. The new operator we describe in this paper can be viewed as the 0–1 split closure of an extended LP formulation that is of polynomial size, and it is therefore possible to optimize over its 0–1 split closure in polynomial time. The extended LP formulation we use is implicitly constructed by the Lovász–Schrijver and Sherali–Adams operators, and we show that the “strengthening step” of these operators can be interpreted as adding certain 0–1 split cuts to this extended LP formulation. Thus our new operator is stronger than the Lovász–Schrijver operator, and we give upper and lower bounds on its strength: we show that our operator can be stronger than the first-level of the Sherali–Adams hierarchy (which equals the Lovász–Schrijver operator), but is weaker than the second-level of the hierarchy.

In the next section we present some background on split cuts and extended formulations. In Section 3, we describe the Lovász–Schrijver and Sherali–Adams lift-and-project operators, and the

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extended LP formulation implicitly constructed by these operators. We define our new lift-and-project operator in terms of the 0–1 split closure of this extended LP formulation, and show that one can optimize over the resulting set of points in polynomial time, as in the case of the Lovász–Schrijver operator. We also show that the second-level of the Sherali–Adams hierarchy is stronger than this new operator. Finally in Section 4, we apply our operator to the stable set polytope and perform numerical experiments to compare its computational performance with that of the Lovász–Schrijver and Sherali–Adams operators.

2. Preliminaries

We use $\mathbb{R}^n$ and $\mathbb{Z}^n$, respectively, for the set of n-dimensional real and integer vectors. Throughout the paper, we work with 0–1 mixed-integer sets of the form $P^{IP} = P \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$ where

$$P = \{x \in \mathbb{R}^n : Ax \le b\},$$

$n = n_1 + n_2$ and $n_1 > 0$, and the inequality system $Ax \le b$ contains the inequalities $0 \le x_i \le 1$ for all $i = 1, \dots, n_1$. We refer to P as the LP relaxation of P^{IP}.

2.1. Extended LP formulations

Let $Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^q : Cx + Dy \le g\}$ be a polyhedron, and let the number of inequalities in $Cx + Dy \le g$ be t. Q is called an *extended formulation* of P if

$$P = \text{proj}_x(Q),$$

where $\text{proj}_x(Q)$ stands for the orthogonal projection of Q onto the space of x variables. More precisely,

$$\text{proj}_x(Q) = \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^q \text{ s.t. } (x, y) \in Q\}.$$

Alternatively, $\text{proj}_x(Q) = \{x \in \mathbb{R}^n : u^T Cx \le u^T g, \ \forall u \in U\}$, where $U = \{u \in \mathbb{R}^t : u^T D = 0, \ u \ge 0\}$ is the *projection cone* of Q. This immediately implies that the projection of a polyhedron is a polyhedron. For more properties of projection, we refer the reader to Balas (2005). Throughout the paper we call Q an *extended LP formulation* of P^{IP} as it is an extended formulation of its LP relaxation.

2.2. 0–1 Split cuts

For a given $i \in I = \{1, \dots, n_1\}$, consider the *0–1 split set*

$$S_i = \left\{x \in \mathbb{R}^n \ : \ 0 < x_i < 1\right\},$$

and note that S_i has an empty intersection with $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$. The inequality $c^T x \ge d$, where $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$, is called a *0–1 split cut* for P generated by S_i if it is valid for $\text{conv}(P \backslash S_i)$. Here we use $\text{conv}(\cdot)$ to denote the convex hull operator. In other words, $c^T x \ge d$ is a 0–1 split cut if it is valid for the *disjunction* $P \cap \{x \in \mathbb{R}^n : x_i = 0\}$ or $P \cap \{x \in \mathbb{R}^n : x_i = 1\}$. A split cut is called *nondominated* if it is not implied by a collection of other split cuts. We note that multiple nondominated split cuts may be generated by the same split set S_i. We denote the *split closure of P with respect to* 0–1 split cuts (or the *0–1 split closure* for short) as

$$\mathcal{S}(P) = \bigcap_{i \in I} \text{conv}(P \setminus S_i).$$

Clearly, $P^{IP} \subseteq \mathcal{S}(P) \subseteq P$. $\mathcal{S}(P)$ is a relaxation of the general split closure defined in Cook, Kannan, and Schrijver (1990) where more general two-term disjunctions are used to generate cuts.

Furthermore, as $\mathcal{S}(P)$ is a polyhedron, it is possible to repeat this operation. For any given integer $k \ge 0$, the k^{th} *0–1 split closure* of P, denoted as $\mathcal{S}^k(P)$, is defined iteratively as follows: $\mathcal{S}^0(P) = P$ and $\mathcal{S}^k(P) = \mathcal{S}(\mathcal{S}^{k-1}(P))$ for $k \ge 1$. Balas et al. (1993) proved that it is sufficient to repeat this operation n_1 times to obtain the convex hull of P^{IP}, establishing that $\mathcal{S}^{n_1}(P) = \text{conv}(P^{IP})$.

We also note that $\mathcal{S}(P)$ can be explicitly written as the projection of an extended formulation using Balas' result on convex hulls of unions of polyhedra (Balas, 1985, Thm 3.3) as follows:

$$\mathcal{S}(P) = \Big\{x \in \mathbb{R}^n \ : \ \exists\ \bar{x}^i, \bar{\bar{x}}^i \in \mathbb{R}^n, \ \bar{\lambda}^i, \bar{\bar{\lambda}}^i \in \mathbb{R}_+ \quad \text{s.t.} \tag{1}$$
$$x = \bar{x}^i + \bar{\bar{x}}^i, \quad \bar{\lambda}^i + \bar{\bar{\lambda}}^i = 1, \quad i \in I,$$
$$A\bar{x}^i \le \bar{\lambda}^i b, \quad A\bar{\bar{x}}^i \le \bar{\bar{\lambda}}^i b, \quad i \in I,$$
$$\bar{x}^i_i = 1, \quad \bar{\bar{x}}^i_i = 0 \quad i \in I\Big\},$$

where $\mathbb{R}_+$ denotes the set of nonnegative real numbers. Note that formulation (1) is of polynomial size (in the encoding size of P) and therefore one can optimize a linear function over it in polynomial time. However, describing the projected set $\mathcal{S}(P)$ by an explicit list of linear inequalities in $\mathbb{R}^n$ may require an exponential number of inequalities. An important point we wish to emphasize is that in contrast to $\mathcal{S}(P)$, it is NP-hard to optimize over the general split closure of P.

Given an extended formulation $Q \subseteq \mathbb{R}^{n+q}$ of P, the 0–1 split closure of Q is defined as

$$\mathcal{S}(Q) = \bigcap_{i \in I} \text{conv}(Q \setminus S_i^+),$$

where $S_i^+ = S_i \times \mathbb{R}^q$. Clearly $\mathcal{S}(Q)$ can also be explicitly written via an extended formulation similar to (1). Therefore, optimization over $\mathcal{S}(Q)$ can also be done in polynomial time provided that Q is of polynomial size.

2.3. Strengthening extended LP formulations with 0–1 split cuts

In terms of optimizing a linear function over the mixed-integer set P^{IP}, the extended LP formulation Q does not lead to better bounds than the original LP relaxation P as

$$\max\{c^T x : x \in P\} = \max\{c^T x : (x, y) \in Q\}$$

for any $c \in \mathbb{R}^n$. However, after the addition of split cuts, extended LP formulations might yield better bounds.

In Bodur et al. (2014), we show that for any split set, and in particular for any 0–1 split set S_i,

$$\text{proj}_x\big(\text{conv}(Q \setminus S_i^+)\big) = \text{conv}(P \setminus S_i).$$

However,

$$\text{proj}_x(\mathcal{S}(Q)) \subseteq \mathcal{S}(P),$$

and the inclusion above is strict in some cases. Therefore adding split cuts to an extended LP formulation can lead to strictly better relaxations than adding split cuts to the original LP relaxation. This, however, can happen only if split cuts generated by multiple split sets are used simultaneously.

Later, in Bodur et al. (2015), we show that there exists an extended LP formulation $Q^* \subseteq \mathbb{R}^{n+n_1-1}$ of the set P^{IP} such that the 0–1 split closure of Q^* is integral, that is,

$$\text{proj}_x(\mathcal{S}(Q^*)) = \text{conv}(P^{IP}).$$

Even though the proof of this result is constructive, it requires an inner description of the set P (i.e., all its extreme points and rays) which is usually difficult to compute from the inequality description of P, i.e., from $Ax \le b$. Moreover, the number of extreme points and rays of P may be exponential in the encoding size of P. Consequently, the approach described in Bodur et al. (2015) is not practical. In the next section we consider extended LP formulations that can be obtained in polynomial time from $Ax \le b$.

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