European Journal of Operational Research 000 (2016) 1–10

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Discrete Optimization

Polynomially solvable cases of the bipartite traveling salesman problem☆

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ARTICLE INFO

Article history:
Received 17 November 2015
Accepted 31 July 2016
Available online xxx

Keywords:
Traveling salesman
Combinatorial optimization
Bipartite traveling salesman
quadrangle property
Kalmanson matrix

ABSTRACT

Given two sets, R and B, consisting of n cities each, in the bipartite traveling salesman problem one looks for the shortest way of visiting alternately the cities of R and B, returning to the city of origin. This problem is known to be NP-hard for arbitrary sets R and B. In this paper we provide an $O(n^6)$ algorithm to solve the bipartite traveling salesman problem if the quadrangle property holds. In particular, this algorithm can be applied to solve in $O(n^6)$ time the bipartite traveling salesman problem in the following cases: $S = R \cup B$ is a convex point set in the plane, $S = R \cup B$ is the set of vertices of a simple polygon and $V = R \cup B$ is the set of vertices of a circular graph. For this last case, we also describe another algorithm which runs in $O(n^2)$ time.

1. Introduction

1.1. Background and prior work

In the traveling salesman problem (TSP), given a prescribed set of cities, one wishes to find the shortest route starting and finishing at the same location, visiting each one of the cities exactly once. This problem is perhaps one of the most extensively studied problems in combinatorial optimization and its different variants have many applications in different areas of knowledge, including computer science, operations research, genetics, engineering and electronics. The reader is referred to Gutin abd and Punnen (2007) and Lawler, Lenstra, Rinooy, and Shmoys (1985) for a review of the state of the art on this problem.

One of these variants is the bipartite TSP (BTSP). Now, the set of cities is partitioned into two classes R and B, with $|R| = |B| = n$, and one wishes to find a shortest route such that the cities in R and B alternate along the route. Besides being interesting in itself, the BTSP is related to other problems, mainly pickup and delivery problems, such as the pick-and-place robots problem (or the printed circuit board assembly problem) (Srivastav, Schroeter, & Michel, 2001), the k-delivery TSP (Anily & Bramel, 1999) or the swapping problem (Anily & Hassin, 1992). For an overview on pickup and delivery problems, see Berbeglia, Cordeau, Gribkovskaia, and Laporte (2007).

In the k-delivery TSP (k-DTSP), one looks for the shortest route to pick up n items located at n source points and deliver them to n sink points, using a single vehicle of capacity k and assuming that an item at a source point can be delivered to any sink point. The BTSP is a special case of the k-DTSP if $k = 1$. In the swapping problem, the main goal is to find the shortest route to swap n objects of $m \leq n$ different types between n locations using a single vehicle with unit capacity. Every location is associated with two object types – the one it currently holds and the one it demands – and holds or demands at most one unit of an object. Moreover, the total supply in the network for each object type equals its total demand. When there are only two object types, the swapping problem is equivalent to the BTSP. The reader is referred to Anily and Bramel (1999), Bhattacharya and Hu (2012), Chalasani and Motwani (1999), Wang, Lim, and Xu (2006) and Anily, Gendreau, and Laporte (1999), Anily, Gendreau, and Laporte (2011), Anily and Hassin (1992), Anily and Pfeffer (2013) for different results and variants on the k-DTSP and the swapping problem, respectively.

In the Euclidean BTSP, the cities are assumed to be points in the plane and the distance between any two points is the Euclidean distance. It is well-known that the BTSP and the Euclidean BTSP are NP-hard, so there is no polynomial algorithm to solve them unless $P = NP$. Moreover, the problem remains NP-hard even in the case of a grid graph. In general, researchers have focused on designing good approximation algorithms. We refer the reader to Anily and Hassin (1992), Baltz and Srivastav (2005), Chalasani, Motwani, and Rao (1996), Frank, Korte, Triesch, and Vygen (1998),

☆ Partially supported by Gobierno de Aragón under Grant E58 (ESF) and projects MTM2012-30951(MINECO) and MTM2015-63791-R (MINECO/FEDER).

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http://dx.doi.org/10.1016/j.ejor.2016.07.060
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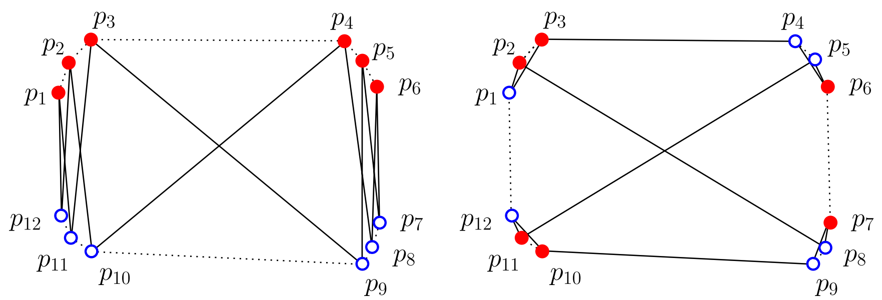

Fig. 1. Left: The shortest Hamiltonian alternating cycle is the criss-cross cycle. Right: The shortest Hamiltonian alternating cycle for another partition of the same convex point set.

Shurbevski, Nagamochi and Karuno (2014), Srivastav et al. (2001) and the references therein for different approximation algorithms along with experimental results. The best known approximation factor for the Euclidean BTSP is 2 (Frank et al., 1998; Chalasani et al., 1996).

There are also some publications in the literature devoted to solving particular cases of the BTSP. In Wang et al. (2006), the authors study the k-DTSP for path and tree graphs. In the case of a path, they give an $O(n^2/\min\{k, n\})$ algorithm for arbitrary k and linear algorithms for $k = 1$ and $k = \infty$. In the case of a tree (see also Anily et al., 2011), they propose an $O(n^2)$ algorithm for $k = 1$ and an $O(n)$ algorithm for $k = \infty$, and show that the problem becomes NP-hard in strong sense if k is arbitrary. In Bhattacharya and Hu (2012), a linear-time algorithm is described to solve the k-DTSP on a path.

Another particular case of the BTSP studied in the literature is related to the shoelace problem (Halton, 1995). In this problem, the objective is to find an optimal strategy for lacing shoes such that the amount of shoelace used is minimized. When the eyelets are arranged in two horizontal lines and two eyelets on the same line are not connected consecutively, then the shoelace problem is an instance of the BTSP (the eyelets placed on the two lines correspond to R and B, respectively). In this case, Halton (1995) proved that the optimal way of threading the shoelaces is the so-called criss-cross lacing strategy, which corresponds to the typical method used in the USA for lacing shoes: threading the shoelaces in opposing zigzags, so that they seem to be crossed when seen from above.

Halton's result was generalized later in Misiurewicz (1996) and Deineko and Woeginger (2014). In both papers the authors show that the criss-cross strategy is still the best way of visiting the cities under certain constraints on the distance matrix D. If $1, 2, \ldots, n$ and $n+1, n+2, \ldots, 2n$ are the cities in R and B, respectively, Misiurewicz (1996) shows that it is sufficient for the distance matrix D to satisfy: $d(i, j) + d(k, l) \leq d(i, l) + d(k, j)$ for $1 \leq i \leq k \leq n$ and $n+1 \leq j \leq l \leq 2n$. In the second paper Deineko and Woeginger (2014) prove the result for a relaxation on the Monge inequalities for a matrix M and provide an $O(n^4)$ algorithm to decide whether there is a renumbering of the cities such that the resulting distance matrix satisfies this relaxation.

1.2. The quadrangle property

A classic example where Misiurewicz's conditions are satisfied is the following. Consider the set of $2n$ vertices of a convex polygon and suppose that the clockwise order of the vertices is $\{1, 2, \ldots, 2n\}$. Assume that the vertices from 1 to n belong to R, the vertices from $n+1$ to $2n$ belong to B, and that the cost of an edge connecting one vertex to another is the Euclidean distance. The well-known *quadrangle property* for a convex quadrilateral states that the total length of the diagonals of the quadrilateral is always bigger than the total length of two opposite sides. In particular, given any four vertices $i < k < j < l$, with i and k belonging to R and j and l to B, the total length of the two crossing edges (i, j) and (k, l) is always bigger than the total length of the two non-crossing edges (i, l) and (k, j). These are Misiurewicz's conditions for reversing the order of the vertices of B. Therefore, the shortest way of visiting alternately the vertices in R and B is the criss-cross cycle, as is shown in the left part of Fig. 1. Vertices belonging to R are illustrated as solid red points and vertices belonging to B as hollow blue points.

Assume now that the $2n$ vertices of the convex polygon are divided into two arbitrary sets R and B of equal size, as in the right part of Fig. 1. Misiurewicz's inequalities are no longer satisfied because the vertices in R and B are not consecutive in the cyclic order. However, it is still true that if two edges (segments) of the bipartite graph defined by R and B cross, then they can be replaced by two other edges (segments) of the bipartite graph, reducing the total length. Using this fact, one can still compute the shortest Hamiltonian cycle C visiting alternately the vertices in R and B, as the right part of Fig. 1 shows.

These are precisely the types of particular cases of the BTSP we study in this paper: instances in which, given a cyclic order on the cities, two "crossing edges" can be replaced by two "non-crossing edges" without increasing the length. This concept is formalized for graphs as follows.

Definition 1. Let $G = (V, E)$ be an undirected graph on the set of vertices $V = \{1, 2, \ldots, N\}$. For an edge $e = (i, j)$ of E, let $d(i, j)$ be the cost of e. Assuming that $(1, 2, \ldots, N)$ is a cyclic order of the vertices of G, we say that G satisfies the quadrangle property if

$$d(i_1, i_4) + d(i_2, i_3) \leq d(i_1, i_3) + d(i_2, i_4)$$

for any four vertices i_1, i_2, i_3, i_4 such that (i_1, i_4), (i_2, i_3), (i_1, i_3), (i_2, i_4) are edges of E and $i_1 < i_2 < i_3 < i_4$ cyclically.

These inequalities, $d(i_1, i_4) + d(i_2, i_3) \leq d(i_1, i_3) + d(i_2, i_4)$, are usually called quadrangle inequalities. Misiurewicz's conditions correspond to the quadrangle property for the particular case of the complete bipartite graph $G = (R \cup B, E)$, with $R = \{1, 2, \ldots, n\}$, $B = \{n+1, n+2, \ldots, 2n\}$ and the cyclic order $(1, 2, \ldots, n, 2n, 2n-1, \ldots, n+1)$.

1.3. Our main contribution

In this paper, we study the BTSP for a complete bipartite graph $G = (R \cup B, E)$ satisfying the quadrangle property, that is, assuming that $(1, 2, \ldots, 2n)$ is a cyclic order of the vertices of G, inequality $d(i_1, i_4) + d(i_2, i_3) \leq d(i_1, i_3) + d(i_2, i_4)$ holds for any four vertices $i_1, i_2 \in R$, and $i_3, i_4 \in B$ such that $i_1 < i_2 < i_3 < i_4$ cyclically. To the best of our knowledge, this problem has only been solved when $R = \{1, 2, \ldots, n\}$ and $B = \{n+1, n+2, \ldots, 2n\}$ (Misiurewicz, 1996). We show that there is a shortest cycle for the BTSP in G not containing a *five-point star* (defined later). Then, we provide an $O(n^6)$