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Optimal allocation policy of one redundancy in a n-component series system

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ABSTRACT

It is of great importance to optimize system performance by allocating redundancies in a coherent system in reliability engineering and system security. In this paper, we focus on the problem of how to optimally allocate one active [standby] redundancy in a n-component series system in the sense of stochastic ordering. For the active case, it is showed that allocating the redundancy to the relatively weaker component leads to longer system's lifetime in the likelihood ratio and reversed hazard rate orders, respectively. For the standby case, we show that the redundancy should be allocated to the weakest component of the series system in the likelihood ratio order. Based on these results, two optimal allocation policies are proposed. Also, some numerical examples are presented to explicate the theoretic results established here.

1. Introduction

The reliability level of a coherent system can be enhanced by adding redundancies to the original system, ant this topic is of great interest and importance in reliability engineering and system security. In real world, two most commonly used types of redundancies are active (or hot) redundancy and standby (or cold) redundancy. In the active case, available spares are used in parallel with the original components of the system and function simultaneously with those components; in the standby case, spares are attached to the components of the system in such a way that each spare functions only after the failure of its corresponding original component. For these two kinds of allocation ways, one is effectively able to measure the performances through stochastic comparisons between the lifetimes of the resulting systems in the sense of various stochastic orders. Many researchers have paid their attentions on this topic in the past decades; see, for example, Boland, El-Neweihi, and Proschan (1988, 1992), Shaked and Shanthikumar (1992), Singh and Misra (1994), Singh and Singh (1997), Valdés and Zequeira (2003, 2004, 2006), da Costa Bueno (2005), da Costa Bueno and do Carmo (2007), Li and Hu (2008), Hu and Wang (2009), Valdés, Arango, and Zequeira (2010), Brito, Zequeira, and Valdés (2011), Misra, Dhariyal, and Gupta (2009), Misra, Misra, and Dhariyal (2011a,b), Li, Yan, and Hu (2011), Zhao, Chan, and Ng (2012), Belzunce, Martinez-Puertas, and Ruiz (2013), Zhao, Zhang, and Li (2015), and Caserta and Voss (2015) and the references therein.

We first recall some pertinent definitions of stochastic orders that will be used in the sequel. Throughout this paper, the term *increasing* is used for monotone *non-decreasing* and *decreasing* is used for monotone *non-increasing*. Let X and Y be two random variables with common support $\Re_+ = [0, \infty)$, density functions f_X and f_Y, distribution functions F_X and F_Y, respectively. Then, $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$ are the survival functions of X and Y, respectively. Denote by $h_X = f_X/\overline{F}_X$ and $h_Y = f_Y/\overline{F}_Y$ the hazard rate functions of X and Y, and $r_X = f_X/F_X$ and $r_Y = f_Y/F_Y$ the reversed hazard rate functions of X and Y, respectively. X is said to be smaller than Y in the *usual stochastic order* (denoted by $X \leq_{\rm st} Y$) if $\overline{F}_X(x) \leq \overline{F}_Y(x)$ for all $x \in \Re_+$; X is said to be smaller than Y in the *hazard rate order* (denoted by $X \leq_{\rm hr} Y$) if $\overline{F}_Y(x)/\overline{F}_X(x)$ is increasing in $x \in \Re_+$, or $h_X(x) \geq h_Y(x)$ for all $x \in \Re_+$; X is said to be smaller than Y in the *reversed hazard rate order* (denoted by $X \leq_{\rm rh} Y$) if $F_Y(x)/F_X(x)$ is increasing in $x \in \Re_+$, or $r_X(x) \leq r_Y(x)$ for all $x \in \Re_+$; X is said to be smaller than Y in the *likelihood ratio order* (denoted by $X \leq_{\rm lr} Y$) if $f_Y(x)/f_X(x)$ is increasing in $x \in \Re_+$. For a comprehensive discussion on various stochastic orders, one may refer to Shaked and Shanthikumar (2007).

Let independent random variables X_1, X_2 and X be the lifetimes of the components C_1, C_2 and redundancy R, respectively. Assume that S is a two-component series system with its components C_1

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and C_2, and a natural question is how to allocate the redundancy R so that the resulting system performs better. In active redundancy case, we want to compare

$$U_1 = \wedge(\vee(X_1, X), X_2) \quad \text{and} \quad U_2 = \wedge(X_1, \vee(X_2, X)),$$

where the symbols "$\wedge$" and "$\vee$" mean min and max, respectively. In standby redundancy case, we want to compare the lifetimes

$$W_1 = \wedge(X_1 + X, X_2) \quad \text{and} \quad W_2 = \wedge(X_1, X_2 + X).$$

In the literature, there has been some work treating this topic. For instance, in the active case, Boland et al. (1992) proved in their seminal work that

$$X_1 \le_{\rm st} X_2 \Longleftrightarrow U_1 \ge_{\rm st} U_2. \tag{1}$$

Singh and Misra (1994) showed that, if X_1, X_2 and X in (1) have exponential lifetimes with hazard rates λ_1, λ_2 and λ, then

$$\lambda_1 \ge \vee\{\lambda_2, \lambda\} \Longrightarrow U_1 \ge_{\rm hr} U_2. \tag{2}$$

Recently, Zhao et al. (2012) strengthened (2) as

$$\lambda_1 \ge \vee\{\lambda_2, \lambda\} \Longrightarrow U_1 \ge_{\rm lr} U_2. \tag{3}$$

You and Li (2014) generalized the result in (3) to the proportional hazard rates (PHR) model.

In the standby case, Boland et al. (1992) showed that

$$X_1 \le_{\rm hr} X_2 \Longrightarrow W_1 \ge_{\rm st} W_2, \tag{4}$$

which was improved by Zhao et al. (2012) under the exponential setup, i.e., if X_1, X_2 and X have exponential distributions with parameters λ_1, λ_2 and λ, then

$$\lambda_1 \ge \lambda_2 \Longrightarrow W_1 \ge_{\rm lr} W_2. \tag{5}$$

However, the result in (5) does not hold for the PHR models as pointed out through a counterexample in You and Li (2014).

It should be noticed that all the results in (1)–(5) mentioned above are limited in the case of two-component series system. Inspired by this, in this paper, we will further pursue the problem of optimal allocation of one active [standby] redundancy in a series system consisting of more than two components. It should be also mentioned here that Boland et al. (1992), Valdés and Zequeira (2006), Misra, Misra, and Dhariyal (2011b) presented several results for n-component series and parallel systems, but here our research methods seem to be quite different from them.

Let $X_1, X_2, \ldots, X_n$ and X be independent exponential random variables with respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ and λ representing the lifetimes of the components $C_1, C_2, \ldots, C_n$ and redundancy, respectively. In the active case, we would like to stochastically compare

$$U_i = \wedge(X_1, \ldots, X_{i-1}, \vee(X_i, X), X_{i+1}, \ldots, X_n)$$

and

$$U_j = \wedge(X_1, \ldots, X_{j-1}, \vee(X_j, X), X_{j+1}, \ldots, X_n),$$

for $1 \le i \ne j \le n$ and $n \ge 2$. In the standby case, we would like to stochastically compare

$$W_i = \wedge(X_1, \ldots, X_{i-1}, X_i + X, X_{i+1}, \ldots, X_n)$$

and

$$W_j = \wedge(X_1, \ldots, X_{j-1}, X_j + X, X_{j+1}, \ldots, X_n).$$

In this regard, we prove for the active redundancy that

$$\lambda_i \ge \vee\{\lambda_j, \lambda\} \Longrightarrow U_i \ge_{\rm lr} U_j \tag{6}$$

and

$$\lambda \ge \lambda_i \ge \lambda_j \Longrightarrow U_i \ge_{\rm rh} U_j. \tag{7}$$

Combining (6) with (7), an equivalent characterization can be derived as

$$\lambda_i \ge \lambda_j \Longleftrightarrow U_i \ge_{\rm rh} U_j. \tag{8}$$

Also, all the results in (6)–(8) can be extended to the PHR models. For the standby redundancy, we show that

$$\lambda_i \ge \lambda_j \Longleftrightarrow W_i \ge_{\rm lr} W_j. \tag{9}$$

These new results in (6)–(9) generalize and strengthen those in (1)–(5) established earlier in the literature. Most importantly, the results established here can help provide a criterion on how to optimally allocate one redundancy in a n-component series system for the design engineer.

The remainder of the paper is organized as follows. The main results are presented in Section 2, in which Section 2.1 puts forward the optimal strategy for allocating an active redundancy in a n-component series system, and the optimal allocation policy for a standby redundancy in a n-component series system is provided in Section 2.2. Some remarks can be found in Section 3 to conclude this paper. For ease of presentation, the detailed proofs of the theorems are deferred in the appendix.

2. Main results

2.1. Active case

As mentioned earlier, Zhao et al. (2012) improved the results in (1) and (2) established by Boland et al. (1992), Singh and Misra (1994) under the exponential framework in the likelihood ratio order. However, all these existing results are based on the framework of the two-component series system. Motivated by this, here we aim to present some general results in a n-component series system instead.

Theorem 2.1. *Let $X_1, X_2, \ldots, X_n$ and X be independent exponential components having respective hazard rates $\lambda_1, \lambda_2, \ldots, \lambda_n$ and λ. Denote*

$$U_i = \wedge(X_1, X_2, \ldots, X_{i-1}, \vee(X_i, X), X_{i+1}, \ldots, X_n)$$

and

$$U_j = \wedge(X_1, X_2, \ldots, X_{j-1}, \vee(X_j, X), X_{j+1}, \ldots, X_n),$$

where $1 \le i \ne j \le n$. Then,

$$\lambda_i \ge \vee(\lambda_j, \lambda) \Longrightarrow U_i \ge_{\rm lr} U_j.$$

Assume that there exists a series system with n exponential components having respective hazard rates $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. Based on this assumption, we know that component 1 performs the worst and component n performs the best. Now, we have an active redundancy (whose lifetime is exponentially distributed with parameter λ) at hand for use. The problem is how should we place the redundancy so that the resulting system could perform the best. If there exists some $k \in \{1, 2, \ldots, n\}$ satisfying that $\lambda_k \ge \lambda \ge \lambda_{k+1}$, upon using Theorem 2.1, we know the redundancy should be allocated in parallel with the component X_1 among $\{X_1, \ldots, X_n\}$ in the likelihood ratio order. However, for the case when $\lambda > \lambda_1$, we cannot figure out the optimal allocation policy according to Theorem 2.1.

To illustrate the theoretical results established in Theorem 2.1, we give the following numerical example.

Example 2.2. Set $\lambda_1 = 4$, $\lambda_2 = 3$ and $\lambda_3 = 2$ in Theorem 2.1. We have the following four cases with different values of λ, wherein allocation policies can be compared in the likelihood ratio order according to Theorem 2.1 except for the last case.