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Continuous Optimization

Computing equilibrium prices for a capital asset pricing model with heterogeneous beliefs and margin-requirement constraints

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ABSTRACT

The mean-variance capital asset pricing model (CAPM) is a useful mathematical tool for studying a variety of financial problems. In contrast to existing work in the literature, which has primarily focused on deriving analytical solutions under restrictive assumptions, we propose a numerical algorithm for efficiently computing the set of equilibrium prices of a CAPM model with heterogeneous investors and arbitrary margin requirements. We present the mathematical formulation of the CAPM model, derive structural properties of the portfolio selection and excess demand functions, and establish the asymptotic convergence of the proposed algorithm under mild conditions. To illustrate the utility of the algorithm, we perform sensitivity analysis on a simple example to study the impact of marginal requirements and interest rates on the resulting equilibrium prices. Numerical studies are also carried out to compare the performance of the algorithm with that of two other popular methods, namely, the fixed point method and the brand-and-bound algorithm.

1. Introduction

The mean-variance capital asset pricing model (CAPM) is a valuable tool in finance. Although its drawbacks and limitations have been evidenced in empirical studies, CAPM remains popular both in theory and practice due to its simplicity and utility in analyzing a variety of financial problems. The shortcomings of the classical CAPM are mostly related to the requirements that all investors have homogeneous expectations and the underlying market is completely efficient—two idealized assumptions that are unlikely to be met in realistic financial markets. Lintner (1969) was the first to investigate the impact of heterogeneity of investors on CAPM by keeping all other assumptions unchanged. Since then, many studies proceeded along the same line; e.g., Levy (1978), Levy, Levy, and Benita (2006), Fama and French (2007), Chiarella, Dieci, He, and Li (2013) and Shi (2016). One of the major factors that may lead to an inefficient market results from short sale restrictions imposed by the government aiming at stabilizing the volatility of asset prices. In particular, since the outbreak of the financial crisis started in 2007, the market has been seeking to restrict short sales to a greater extent by charging margin requirements, and lenders such as brokage firms usually collect a certain amount of capital from borrowers to prevent themselves from potential losses caused by adverse market movements. This has led to studies focusing on using leverage-constrained models to examine the effect of margin requirements on equilibrium asset prices and trading volumes. For example, Lam, Yu, and Lee (2010) propose a margin scheme on when to change the margin requirement and to what level it should be adjusted. Gârleanu and Pedersen (2011) show that margin requirements may cause securities with identical cash flows to be traded at different prices, leading to price gaps and thus violating the law of one price. Anufriev and Tuinstra (2013) conclude that margin requirements will not change the stability of the market equilibrium; however, when some assets are overvalued, the costs of margin requirements may result in increased mispricing and price volatility. More recently, Rytchkov (2014) considers endogenous time-varying margin requirements and analyzes how these requirements affect the capital market equilibrium. Ma, Hu, and Xu (2016) demonstrate that the well-known "Lintnerian invariance" of short selling, i.e., short selling has no impact on asset prices and trading volumes, may no longer hold when there are heterogeneous margin rules.

Existing studies on pricing and trading effect of margin rules typically require obtaining an optimal solution to the CAPM model or its variants. This is usually done analytically by differentiating each individual's portfolio selection problem and then deriving explicit expressions for the equilibrium prices based on market

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clearing conditions, see, e.g., Lintner (1969), Levy (1978), Chiarella et al. (2013), and He and Shi (2007). However, in the presence of additional margin requirements, the budget constraints for the portfolio selection problems become nonlinear (e.g., Gârleanu & Pedersen, 2011). Although in principle, the resulting problems can still be solved by using the KKT conditions, checking these conditions requires enumerating all possible scenarios when inequality constraints become active, which is only analytically tractable in very simple cases when the size of the problem is small (e.g., Jarrow, 1980). For richer and larger models, finding the equilibrium prices will generally need to be carried out using numerical methods.

There is an abundant body of literature on computational approaches for solving general economic equilibrium problems. Most of these methods rely on optimization theory, including the use of mathematical programming with equilibrium constraints (Luo, Pang, & Ralph, 1996), the variational inequality approach (Dafermos, 1990; Jofré, Rockafellar, & Wets, 2007), and algorithms specifically developed for models with linear utility functions (e.g., Garg & Kapoor, 2006; Ghiyasvand & Orlin, 2012). The basic idea is to formulate an equilibrium model as a complementarity optimization problem so that standard optimization tools such as gradient descent and branch-and-bound can be applied. The fixed point method, pioneered by Scarf and Hansen (1973), is another major class of iterative algorithms for approximating economic equilibria; see also Eaves (1972) and Eaves and Schmedders (1999). In contrast to mathematical programming, a salient feature of fixed point algorithms is that they are often globally convergent regardless of the choice of initial solutions. Although some of these approaches can also be applied to determine the equilibrium prices of CAPM models, it is well known that gradient-based methods will in general lead to locally optimal solutions, whereas popular approaches for problems with complementarity constraints, such as branch-and-bound, may have a worst case complexity that is as high as that of exhaustive search. On the other hand, the fixed point method often involves a discretization of the underlying problem, which may lead to computational difficulties, either resulting in a solution space that is too large or in a solution that is not accurate enough.

In this paper, we propose a derivative-free recursive algorithm for computing the equilibrium price of a CAPM model with heterogeneous investors and arbitrary margin requirements. In particular, we focus on the model proposed in Ma et al. (2016), which falls into the general mean-variance framework of Markowitz (1952). The model has essentially the same setup as that of the early work by Jarrow (1980) except that it allows for additional heterogeneous margin rules for short selling. In addition, it is also consistent with the dynamic models proposed in Gârleanu and Pedersen (2011), Anufriev and Tuinstra (2013), and Rytchkov (2014) when restricted to single periods. Thus, the primary reason for considering such a model is that it is typical of a number of alternatives formally investigated in the literature, so that solutions obtained and new insights gained from the model could be easily compared and contrasted with existing findings in the field.

The idea underlying our proposed algorithm is based on the intuition that an excess of demand over supply of a certain asset will lead to an increase in its price, whereas an excess supply would tend to lower the price. Therefore, at each iteration of the recursive procedure, the price of each tradable asset is incremented by a small amount proportional to the difference between the market demand and supply of the same asset. The iteration process continues until little or no further improvement can be made, in which case an approximation of the equilibrium price is obtained. The algorithm has a similar flavor as the tâtonnement method proposed by Walras (1954), which has been widely applied to investigate equilibrium stability in economics. However, the convergence of the tâtonnement process assumes that the market satisfies the gross substitutes property (e.g., Arrow & Hurwicz, 1960), a condition that does not hold in our case due to the intricate correlations among risky assets. We justify the asymptotic validity of the algorithm under mild regularity conditions. In particular, our main theoretical contributions are the establishment of the uniform continuity of the optimal portfolio selection function for each individual investor and the (local) monotonicity of the market excess demand function. The latter ensures demands to move in the opposite direction of prices in the vicinity of an equilibrium point, which in turn implies that the sequence of price vectors generated by the algorithm will converge to the set of equilibrium prices.

To illustrate the application of the proposed algorithm, we use it to conduct an empirical sensitivity analysis on a simple example to study the impact of margin requirements and interest rates on the resulting equilibrium prices. The performance of the algorithm is then further tested on a set of randomly generated problem instances and compared with that of two other methods: the fixed point algorithm of Laan and Talman (1979) and the well-known branch-and-bound algorithm. Numerical results indicate that our algorithm scales well with the problem size and is capable of handling relatively large problem instances involving hundreds of investors and tradable assets within a practical amount of time.

The rest of the paper is structured as follows. In Section 2, we describe the CAPM model with heterogeneous margins and list the assumptions imposed on the model. In Section 3, we give a detailed description of the proposed algorithm. In Section 4, we analyze its asymptotic convergence behavior. Computational results are provided in Section 5 to illustrate the empirical performance of the algorithm. Finally we conclude the paper in Section 6.

2. A CAPM model with heterogeneous margin requirements

Suppose there are K investors trading in an economy consisting of $J+1$ tradable securities indexed by $j \in \{0, 1, \ldots, J\}$, where $j = 0$ represents a riskless asset with risk-free interest rate r. We consider a single-period ($t = 0, 1$) model. Let p_j be the price of asset j at $t = 0$ and denote by the random variable X_j the price of asset j at $t = 1$. At the beginning of period $t = 0$, each investor k is endowed with a portfolio that contains $n_j^k \geq 0$ shares of asset j, $j = 0, \ldots, J$. Thus, the aggregate market endowment for security j is $\sum_{k=1}^{K} n_j^k$ and the initial aggregate wealth of investor k is given by $W_0^k = \sum_{j=0}^{J} n_j^k p_j$. We assume that there is no restriction on borrowing and lending of the risk-free asset and there is no transaction cost on any trading. However, short selling risky asset j by investor k will incur a margin requirement of $100 \times m_j^k$ percent of the price of security j.

At $t = 0$, investor k can rebalance his/her portfolio through operations such as shorting and longing on securities under the initial budget constraint to maximize his/her preference at $t = 1$. Let ϕ_j^k be the position of asset j held by investor k after rebalancing. The budget constraint of investor k can be written as follows:

$$W_0^k = \sum_{j=1}^{J} [p_j \phi_j^{k-} m_j^k + p_j \phi_j^{k+}] + \phi_0^k, \tag{1}$$

where $\phi_j^{k-} = \max\{-\phi_j^k, 0\}$, $\phi_j^{k+} = \max\{\phi_j^k, 0\}$. In (1), the term $p_j \phi_j^{k-} m_j^k$ represents the margin requirements incurred when short selling asset j and $p_j \phi_j^{k+}$ is the cost of buying ϕ_j^{k+} shares of asset j. Note that we have normalized p_0 to 1 for simplicity. It can be seen that the wealth of investor k at the end of holding period $t = 1$ becomes

$$W_1^k = W_0^k + r\phi_0^k + \sum_{j=1}^{J} (X_j - p_j)\phi_j^k, \tag{2}$$