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Decision Support

The mathematical equivalence of the "spanning tree" and row geometric mean preference vectors and its implications for preference analysis

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ABSTRACT

Pairwise comparison is a widely used approach to elicit comparative judgements from a decision maker (DM), and there are a number of methods that can be used to then subsequently derive a consistent preference vector from the DM's judgements. While the most widely used method is the eigenvector method, the row geometric mean approach has gained popularity due to its mathematical properties and its ease of implementation. In this paper, we discuss a spanning tree method and prove the mathematical equivalence of its preference vector to that of the row geometric mean approach. This is an important finding due to the fact that it identifies an approach for generating a preference vector which has the mathematical properties of the row geometric mean preference vector, and yet, in its entirety, the spanning tree method has more to offer than the row geometric mean method, in that, it is inherently applicable to incomplete sets of pairwise comparison judgements, and also facilitates the use of statistical and visual techniques to gain insights into inconsistency in the DM's judgements.

1. Introduction

Pairwise comparison (PC) is a widely used approach to elicit comparative judgements from a decision maker (DM). In the PC method, the DM is asked a series of questions to compare the available options in pairs, and eventually, a prioritization method is applied to these judgements in order to estimate the DM's preferences in the form of a preference vector. The preference vector is a vector of weights representing the relative strength of preferences for available options. However, since the judgements acquired from the DM often contain inconsistency, the process of estimating a preference vector is not necessarily straightforward. Inconsistency occurs when the direct comparative value of a pair of options does not match the indirect comparative value derived from an intermediate third option. For example, if option A is declared twice as preferred as option B and option B is declared three times as preferred as option C, then the indirect comparative value suggests that option A be preferred six times more than option C and yet the DM may directly declare option A to be say five times as preferred as option C, which is obviously inconsistent with the other two comparative judgements. That is the direct comparative value of Option A and Option C (i.e. 5) does not match the indirect comparative value of Option A and Option C derived from an intermediate third option B (i.e. 6). Of course, the number of comparisons increases with the number of options which, in turn, increases the possibility of having at least some and possibly a high number of inconsistent comparisons. Therefore, any prioritization method must be able to estimate the preference vector from an inconsistent set of comparisons.

Historically, the principal right eigenvector (REV) prioritization method (Saaty, 1977) has been widely used for estimating the preference vector for both consistent and (acceptably) inconsistent PC judgements where, in the REV method, the PC judgements are used to construct a PC matrix, the principal eigenvector of which is taken as the preference vector. The inconsistency is measured in terms of the Consistency Ratio (CR) which is an Eigenvalue based measure with the PC matrix only considered acceptable if the CR value remains below a certain limit (usually $CR < 0.1$).

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Johnson (1979) discovered that, for the same problem, the use of left eigenvectors may produce a different solution to that of the right eigenvector approach, yet considered the use of left eigenvectors to be as equally justified as the use of right eigenvectors. Therefore, the REV method has been criticized due to this left-right eigenvector asymmetry, the use of arbitrary thresholds for inconsistency acceptability, as well as a few other further issues (Bana e Costa & Vansnick, 2008; Barzilai, 1997; Barzilai, Cook, & Golany, 1987). Due to these shortcomings, several other prioritization methods for preference vector estimation have been proposed in the literature which also begin by constructing a PC matrix from the PC judgements. For example, the logarithmic least squares (LLS) method, proposed in Crawford and Williams (1985), assumes that the most preferred approach for prioritization is to find the vector that minimizes the sum of the logarithmic residuals from a given set of judgements. Considering the multiplicative properties of PC, Crawford and Williams (1985) showed that the LLS method always generates a unique solution, and in the case of a complete set of PC judgements, the LLS solution is identical to the solution calculated using the row geometric mean (RGM) of the constructed PC matrix. In addition to these approaches, there exists a number of other optimization-based methods like direct least squares (DLS) (Chu, Kalaba, & Spingarn, 1979), logarithmic least absolute value (LLAV) (Cook & Kress, 1988), and fuzzy preference programming (Mikhailov, 2000). Choo and Wedley (2004) analysed and numerically compared a variety of these prioritization methods and concluded that there is no single best method that outperforms the others in every situation.

Although REV is the most commonly used method, the RGM approach has gained popularity due to its mathematical properties, and while shown to be equivalent to the LLS approach (Crawford & Williams, 1985), RGM has additional benefits due to its ease of implementation (Crawford, 1987; Williams & Crawford, 1980). Indeed (Williams & Crawford, 1980) proposed using the RGM method rather than the REV method due to its ease of computation, and also demonstrated its advantages arising from common statistical and mathematical properties. Since the objective of the prioritization method is to obtain a single preference vector from an inconsistent PC matrix, most methods therefore justifiably focus on this aspect, and therefore assess inconsistency only by measuring it for the purpose of accepting or rejecting the provided PC judgements as suitable rather than analysing inconsistency. That is, while focusing on this "single solution" aspect, an in-depth analysis of the inconsistency is neglected.

We contend that a prioritization method must have the capabilities to focus on both aspects of the problem, i.e. production of a single "good quality" preference vector and also facilitation of an in-depth inconsistency analysis. The latter aspect is illustrated in Section 4.1 by establishing an underlying universe of potential preference vectors and then examining the degree of homogeneity within them. In this way we can start to unravel any inconsistency in the decision maker's judgements by translating inconsistency into a number of different possible mindsets. This is important particularly of course when inconsistency is high and so where the DM may need significant help to resolve his/her inconsistency, but also sometimes even when CR is low, as situations can arise where even though the CR value might otherwise be regarded as acceptably low, it is clear that using this acceptability criterion may be quite inappropriate - see illustration in Section 4.1.

Also, Harker (1987b) investigated incomplete sets of judgements where the DMs are allowed to respond with "do not know" or "not sure" to some judgements. This is an important issue to investigate as the probability of acquiring an incomplete set of PC judgements increases with an increase in the total number of items for comparison (Fedrizzi & Giove, 2007, 2013; Schubert, 2014). Both the REV and the RGM methods are inappropriate in such cases due to the fact that the PC matrix cannot be constructed without estimating/imputing the missing judgements (see Section 4.2 for details).

Indeed, several criteria have been suggested to compare prioritization methods in the literature. For example, minimal deviation from the DM's judgements (Kou & Lin, 2014; Lin, 2007; Siraj, Mikhailov, & Keane, 2012b), computational complexity, ability to handle incomplete sets of judgements (Ergu, Kou, Peng, Shi, & Shi, 2011; Harker, 1987a; Srdjevic, Srdjevic, & Blagojevic, 2014), adhering to geometric properties (Aguaron & Moreno-Jimenez, 2003; Barzilai, 1997), and ability to measure inconsistency (Brunelli, Canal, & Fedrizzi, 2013; Brunelli & Fedrizzi, 2015; Tomashevskii, 2015). While there is no consensus with regards to which of these "conventional" performance measures should be used for comparative assessment, we contend that a prioritization method should meet as many of these criteria as possible, and must also have the ability to facilitate the analysis of inconsistency.

In this context, a graph-theoretic approach was recently formulated to calculate a preference vector by taking the average of all possible preference vectors calculated through enumeration of all possible spanning trees (EAST) (Tsyganok, 2010; see also Siraj, Mikhailov, & Keane, 2012a). The proposed method was shown to have a number of desirable properties including, for example, producing a solution with minimal deviation from the PC judgements and measuring the level of inconsistency in these judgements. However, since the original method used the arithmetic mean to calculate the average, it failed to satisfy the criterion of adhering to geometric properties. We have therefore investigated the use of the geometric mean of all "spanning tree" preference vectors (GMAST).

In this paper, we report on the quality of the GMAST method's preference vector and its adherence to the conventional performance criteria, and provide some initial insights into its capability to facilitate the analysis of inconsistency. We therefore focus on the GMAST preference vector and prove its mathematical equivalence to that of the RGM method. This is an important finding due to the fact that it establishes the quality of the GMAST preference vector by proving that it has the mathematical properties of the RGM preference vector and yet, the GMAST method in its entirety has additional benefits. That is, unlike RGM, the GMAST method is inherently applicable to incomplete PC matrices (see Section 4.2), and also facilitates in-depth inconsistency analysis (see Sections 4.1 and 6). Indeed, with respect to all of the performance criteria, the GMAST method in its entirety outperforms all the other existing prioritization methods.

2. Problem formulation

Assume that we are interested in determining a preference vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ where $\frac{w_i}{w_j}$ represents the DM's relative preference for element i compared to element j. Because we are only interested in the ratio $\frac{w_i}{w_j}$, $\mathbf{w}$ is not unique and there is a class of equivalent vectors satisfying our requirement where any member of the class only differs from another member by a multiplicative scalar.

Assuming that $A = [a_{ij}]$ is the DM's PC matrix (i.e. a_{ij}= the acquired DM's judgement for element i compared to element j), then the objective of a prioritization method is to derive a $\mathbf{w}$ from A.

Since $a_{ii} = 1$ for all $i = 1, 2, \ldots, n$, we have

$$A = \begin{bmatrix} 1 & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & 1 & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & 1 & \ldots & \ldots \\ \ldots & \ldots & \ldots & 1 & \ldots \\ a_{n1} & a_{n2} & \ldots & \ldots & 1 \end{bmatrix}$$

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