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Innovative Applications of O.R.

Evaluations of quantiles of system lifetime distributions

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ABSTRACT

For each coherent and mixed system with exchangeable components, we provide sharp bounds on the deviations of system lifetime distribution quantiles from the respective quantiles of single component lifetime distributions. The bounds are expressed in the scale units generated by the absolute moments of various orders of the component lifetime centered about the median of its distribution.

1. Introduction

Reliability theory is a branch of applied probability devoted to optimizing the probability of functioning complex technical systems. The classic references on reliability are Barlow and Proschan (1965, 1975). For a more recent account of the theory we refer the reader to Epstein and Weissman (2008) and Aven and Jensen (2013). Two-state coherent systems are the basic objects of reliability analysis. Their operation rules are described by the structure functions φ: $\{0, 1\}^n \mapsto \{0, 1\}$. Here 0 and 1 mean the failure status and working status of each component and the system, and n denotes the number of elements of the system. For given $(x_1, \dots x_n) \in \{0, 1\}^n$, with x_i standing for the working status of ith component, $\varphi(x_1, \dots, x_n)$ informs whether the system works for respective values of $x_1, \dots, x_n$. The system is called coherent if it is monotone, i.e. $x_i \le y_i$, $i = 1, \dots, n$, implies $\varphi(x_1, \dots x_n) \le \varphi(y_1, \dots y_n)$, and all its elements are relevant, i.e. for every $i = 1, \dots, n$ there exist $x_1, \dots x_{i-1}, x_{i+1}, \dots, x_n \in \{0, 1\}$ such that $\varphi(x_1, \dots x_{i-1}, 1, x_{i+1}, \dots, x_n) - \varphi(x_1, \dots x_{i-1}, 0, x_{i+1}, \dots, x_n) = 1$. The monotonicity property is intuitively clear: the system with some extra damaged units cannot function better. The other property says that the system does not contain elements that do not affect its functioning.

It is assumed that the lifetimes $T_1, \dots, T_n$ of system components are random, and so is the system lifetime T. If the system is composed of identical items so that one can interchange their roles in the system without affecting its performance, it is natural to assume that the joint distribution of component lifetimes is exchangeable. In particular, independent identically distributed random variables are exchangeable, but the independence condition is often violated in practice.

Analysis of systems with exchangeable components is greatly simplified with use of Samaniego signature. This is a vector $\mathbf{s} = (s_1, \dots, s_n)$ belonging to the simplex of size n, dependent on the system structure, and defined as follows

$$s_i = \frac{1}{\binom{n}{i-1}} \sum_{\sum_{j=1}^n x_j = n-i+1} \varphi(x_1, \dots x_n) - \frac{1}{\binom{n}{i}} \sum_{\sum_{j=1}^n x_j = n-i} \varphi(x_1, \dots x_n), \quad i = 1, \dots, n. \tag{1}$$

The notion was introduced in Samaniego (1985) for the systems composed of units with i.i.d. continuous lifetimes, and its usefulness for the systems with exchangeable components was proved by Navarro, Balakrishnan, Samaniego, and Bhattacharya (2008). Formula (1) was presented in Boland (2001). The signature does not uniquely determines the system structure, because there are essentially different systems with identical signatures. The Samaniego formula says that the system lifetime distribution is a convex combination of distributions of order statistics based on the component lifetimes, and the coefficients of the combination coincide with the consecutive coordinates of the signature vector. The statement is precisely formulated in Lemma 1 of Section 2.

Since the order statistics of $T_1, \dots T_n$ are the lifetimes of so-called k-out-of-n:F systems, $k = 1, \dots, n$, functioning till the kth failure among n system components, the Samaniego representation asserts that the lifetime distribution of a system with exchangeable components and signature $\mathbf{s} = (s_1, \dots, s_n)$ is identical with a randomly chosen one of k-out-of-n:F systems, $k = 1, \dots, n$,

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with the corresponding choice probability s_k. This was an incentive for Boland and Samaniego (2004) for introducing the notion of mixed systems with signature $\mathbf{s} = (s_1, \ldots, s_n)$ being an arbitrary point of n-dimensional simplex. The mixed systems with signature $\mathbf{s} = (s_1, \ldots, s_n)$ is just a randomly selected k-out-of-n:F system with choice probability distribution represented by the signature vector. The family of mixed systems is more convenient for mathematical analysis than the set of coherent systems. For instance, every coherent and mixed system with $1 \le m \le n$ exchangeable components can be represented as a mixed system of size n.

Under the assumption that $T_1, \ldots, T_n$ are non-degenerate exchangeable random lifetimes of components of a mixed (coherent, in particular) system of size n with signature $\mathbf{s} = (s_1, \ldots, s_n)$, let G stand for the marginal distribution function of a single component lifetime. We also write H for the distribution function of the system lifetime T. Define the left-continuous quantile function of G as

$$G^{-1}(q) = \inf\{x : \; G(x) \ge q\}, \quad 0 < q < 1.$$

By definition, $G^{-1}(q)$ is the smallest quantile of order q. Similarly, we introduce $H^{-1}(q)$, $0 < q < 1$.

The purpose of the paper is to present the sharp lower and upper bounds on

$$\frac{H^{-1}(q) - G^{-1}(q)}{(\mathbb{E}|T_1 - G^{-1}(\frac{1}{2})|^p)^{1/p}} \tag{2}$$

for a system with given signature (suppressed in the notation for brevity) and for arbitrarily fixed $0 < q < 1$ and $p > 0$. In other words, we aim at optimal evaluating the greatest possible deviations of various quantiles of the system lifetime from the respective quantiles of the lifetime of single component measured in the scale units generated by pth absolute moments of the component lifetime centered about its median for various $p > 0$. Fixing p, we tacitly assume that $\mathbb{E}|T_1|^p < \infty$ that assures that the denominator is well defined. By assumption, it is strictly positive as well.

As an immediate consequence, we obtain the upper bounds for the absolute deviations of quantiles of system lifetime $|H^{-1}(q) - G^{-1}(q)|$ gauged in the same scale units $(\mathbb{E}|T_1 - G^{-1}(\frac{1}{2})|^p)^{1/p}$. We similarly derive analogous evaluations for the variation of lifetime quantiles $H_1^{-1}(q) - H_2^{-1}(q)$ of two identical systems under two assumptions on the joint distributions of respective component lifetimes. One is that they all have identical marginal distribution, and different structures of dependence. The other is that they only have identical location and scale measures $G^{-1}(q)$ and $(\mathbb{E}|T_1 - G^{-1}(\frac{1}{2})|^p)^{1/p}$, respectively.

The results of the paper provide extensions of classic investigations on evaluating differences between the expectations of system and component lifetimes $\mathbb{E}T - \mathbb{E}T_1$ expressed in the scale units $(\mathbb{E}|T_1 - \mathbb{E}T_1|^p)^{1/p}$ based on the central absolute moments of various orders of the element lifetime. These are analogous to the bounds on the expectations of convex combinations of order statistics. The justification in the exchangeable case was presented in Navarro and Rychlik (2007). Rychlik (1993b) determined sharp lower and upper bounds on arbitrary linear combinations of order statistics for arbitrarily dependent identically distributed random variables as well as exchangeable variables, expressed by means of various centered absolute moments. Similar results for systems with identical and independent components can be concluded from Rychlik (1998). More refined optimal bounds were established under additional restrictions on the marginal distributions of component lifetimes. E.g., Danielak (2003) and Goroncy and Rychlik (2015a, 2015b) presented such bounds for families of distributions with monotone density and failure rate functions.

Information about various quantiles of system lifetime provides more insight into its nature than the single mean parameter. The value of $H^{-1}(q)$ is just the time period that system survives with probability q. It clearly strongly depends on the quality of system components, and this is the reason that $H^{-1}(q)$ is compared with $G^{-1}(q)$. Moreover, the values of $H^{-1}(q) - G^{-1}(q)$ depend on the time units they are measured in. A trivial notice is that the duration time gauged in minutes is 60 times greater than the time measured in hours. This explains the necessity of dividing $H^{-1}(q) - G^{-1}(q)$ by some scale units. We choose the units based on the component lifetime which are easier to get there. The pth absolute moments centered about the median are the most natural ones. Note that the other popular choices, e.g. moments centered about the mean require finiteness of the first moment of T_1. Applications of $(\mathbb{E}|T_1 - G^{-1}(\frac{1}{2})|^p)^{1/p}$ for $p < 1$ requires only $\mathbb{E}|T_1|^p < \infty$.

The primary problem of our study was to compare the medians of component and system lifetimes gauged in units based on the median of component lifetime. This has a reasonable solution even if $\mathbb{E}T_1^p$ for any $p > 0$ does not exist. It appeared that with the same method one can compare arbitrary quantiles of the distributions. However, our bounds are most useful for central quantiles, because they tend to infinity as q tends to 0 and 1.

Miziuła and Rychlik (2014) provided sharp lower and upper bounds on the ratio $\mathbb{V}ar\, T / \mathbb{V}ar\, T_1$ in our model. It was shown in Miziuła and Rychlik (2015) that these estimates remain valid and optimal for a very wide family of dispersion measures. Here we also point out possibilities of extending our results to evaluations of quantile differences expressed in scale units based on generalized moments of $\left|T_1 - G^{-1}\left(\frac{1}{2}\right)\right|$. However, these bounds strongly depend on the choice of particular scale unit.

Reliability analysis is extensively developed nowadays. More and more complex system structures are studied with stand-by components and modules, and various types of possibly repairable failures. A greater attention is paid to interdependencies among the components and their lifetimes. Chen, Yang, Ye, and Kang (2015) discussed various types of failure mechanisms, their impact on still working items, and resulting system reliability. Another attempt of modelling dependence in reliability was presented by Wu (2014). Coit, Chatwattanasiri, Wattanapongsakorn, and Konak (2015) performed a dependence analysis of k-out-of-n system reliability under assumption that some groups of components necessarily have to work together. A problem of effective allocation of redundant components in systems with dependent component lifetimes was studied by Belzunce, Martinez-Puertas, and Ruiz (2013). Fiondella and Xing (2015) analyzed the effect of component correlation on the system reliability and mean time to failure functions. Gupta, Misra, and Kumar (2015) performed various stochastic ordering comparisons of systems with dependent and identically distributed component lifetimes. Sebastio, Trivedi, Wang, and Yin (2014) proposed an efficient algorithm of calculating network reliability bounds. Strigini and Wright (2014) derived interesting bounds on survival probability under some uncertainty on parameters of reliability functions. Salman, Li, and Steward (2015) treated a practical problem of evaluating the reliability of system subjected to hurricanes. We finally mention two recent applications of system signatures. Coolen and Coolen-Maturi (2015) used them for predicting system reliability whereas Feng, Patelli, Beer, and Coolen (2016) provided some bounds on the survival function and description of component importance.

The rest of the paper is drawn up as follows. Section 2 contains auxiliary definitions and results. They are used in Section 3 for formulating and proving our main results. Their applications to particular systems are presented in Section 4.

2. Auxiliary notions and results

For a fixed signature $\mathbf{s} = (s_1, \ldots, s_n)$, we define two increasing subsequences $(i_0, \ldots, i_K)$ and $(j_0, \ldots, j_M)$ of sequence $(0, \ldots, n)$ with lengths $1 \le K+1, M+1 \le n+1$ dependent on $\mathbf{s}$. For this pur-