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Discrete Optimization

Lower bounds and algorithms for the minimum cardinality bin covering problem

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ABSTRACT

This paper introduces the minimum cardinality bin covering problem where we are given m identical bins with capacity C and n indivisible items with integer weights w_j $(j = 1, \dots, n)$. The objective is to minimize the number of items packed into the m bins so that the total weight of each bin is at least equal to C. We discuss reduction criteria, derive several lower bound arguments and propose construction heuristics as well as a powerful subset sum-based improvement algorithm that is even optimal when $m = 2$. Moreover, we present a tailored branch-and-bound method which is able to solve instances with up to 20 bins and several hundreds of items within a reasonable amount of time. In a comprehensive computational study on a wide range of randomly generated instances, our algorithmic approach proved to be much more effective than a commercial solver.

1. Introduction

1.1. Problem definition

In this paper we address the *minimum cardinality bin covering problem* (MCBCP) which consists in determining the least number of items necessary to fill (or cover) m bins. More precisely, given $m \geq 2$ identical bins of capacity $C \in \mathbb{N}$ and a set $\mathcal{J}$ of n indivisible items with integer weights $w_j \in \mathbb{N}$ $(j = 1, \dots, n)$ the objective is to minimize the number of items packed into the m bins so that the total weight of each bin equals at least C. Introducing binary variables x_{ij} which take the value 1 if item j is packed into bin i and 0 otherwise, a straightforward formulation of the MCBCP as an integer linear program (ILP) consisting of objective function (1) subject to (2)–(4) is provided below.

$$\text{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \tag{1}$$

subject to

$$\sum_{j=1}^{n} w_j x_{ij} \geq C \qquad i = 1, \dots, m \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} \leq 1 \qquad j = 1, \dots, n \tag{3}$$

$$x_{ij} \in \{0, 1\} \qquad i = 1, \dots, m;\ j = 1, \dots, n \tag{4}$$

Objective function (1) minimizes the number of items necessary to fill all m bins. Constraints (2) ensure that each bin is filled and constraints (3) guarantee that each item is assigned to at most one bin. Finally, the domains of the binary variables are set by (4). By reduction from 3-Partition (cf. Garey & Johnson, 1979) it is readily verified that MCBCP is $\mathcal{NP}$-hard. Throughout the paper, we assume the items to be labeled in such a way that $w_1 \geq w_2 \geq \dots \geq w_n > 0$. Moreover, for economy of notation, we often identify items by their index.

As a possible application of MCBCP, consider the disposal or transportation of m different liquids (e.g., chemicals) that cannot be mixed. If at least C volume units of each liquid have to be transported and we are given n tanks of various sizes, the MCBCP is to load the m liquids into the fewest number of tanks. Clearly, the less tanks are used the more convenient the handling and the less organizational effort. Note that here, the "liquids" correspond to bins and the "tanks" (and their sizes) correspond to the items (and their weights). For the closely related *liquid loading problem* we refer to Christofides, Mingozzi, and Toth (1979).

1.2. Related work

Problem MCBCP can be seen as the dual version of the *maximum cardinality bin packing problem* (MCBPP) which consists in

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determining the maximum number of indivisible items that can be packed into the m bins so that the total weight of each bin does not exceed C. The MCBPP has been widely studied in terms of upper bounds and exact solution procedures (Labbé, Laporte, & Martello, 2003; Peeters & Degraeve, 2006), worst-case performance of heuristics (Coffman & Leung, 1979; Coffman, Leung, & Ting, 1978; Langston, 1984), probabilistic analyses (Bruno & Downey, 1985; Foster & Vohra, 1989; Rhee & Talagrand, 1993), meta-heuristics (Loh, Golden & Wasil, 2009), and innovative applications (Vijayakumar, Parikh, Scott, Barnes, & Gallimore, 2013). In contrast to the variety of publications on MCBPP, to the best of our knowledge we are the first to study its dual version MCBCP which has recently been mentioned for the first time by Coffman, Csirik, Galambos, Martello and Vigo (2013) as a natural variant of the bin covering problem (BCP).

The BCP itself has been introduced by Assmann, Johnson, and Kleitman (1984) as the dual version of the classical one-dimensional bin packing problem. So far, related work on variants of the BCP stem from, e.g., Fukunaga and Korf (2007) who considered the problem of minimizing the total cost of the used items to cover m variable sized bins, Csirik, Epstein, Imreh, and Levin (2010) who examined the online and two semi-online versions of the problem of minimizing the total weight of the items used to cover m bins, Epstein, Imreh, and Levin (2010) who investigated a class constrained bin covering problem where each item has a color associated with it and the goal is to cover as many bins as possible subject to the constraint that the total number of distinct colors in each bin has to be at least l, and Epstein, Imreh, and Levin (2013) who studied the cardinality constrained bin covering problem which consists in maximizing the number of covered bins subject to the constraint that each bin must contain at least k items.

Returning to the MCBCP and its ILP formulation, it becomes obvious that MCBCP belongs to the class of mixed packing covering integer programs which are formally defined as:

$$\text{Minimize } z = c^T x \tag{5}$$

subject to

$$Ax \geq a \tag{6}$$

$$Bx \leq b \tag{7}$$

$$x \leq d \tag{8}$$

$$x \in \mathbb{Z}^N_{\geq 0} \tag{9}$$

where $A \in \mathbb{R}^{M\times N}_{\geq 0}$, $B \in \mathbb{R}^{R\times N}_{\geq 0}$, $a \in \mathbb{R}^{M}_{>0}$, $b \in \mathbb{R}^{R}_{\geq 0}$, $c \in \mathbb{R}^{N}_{\geq 0}$, and $d \in \mathbb{R}^{N}_{>0}$ (cf. Kolliopoulos & Young, 2005). The constraints (6), (7), and (8) are called covering, packing, and multiplicity (or capacity) constraints, respectively. Note that the multiplicity constraints can also be modeled by adding (at most) N rows to B – one for each constraint $x_j \leq d_j (<\infty)$. This equivalent notation appeared in Kolliopoulos and Young (2001) under the name Covering Integer Problems (CIP) with generalized multiplicity constraints. However, it is also important to note that a packing constraint cannot be multiplied by -1 in order to be turned into a covering constraint because the problem definition presupposes non-negative data.

Setting $M = m$, $R = n$, $N = mn$, $a = (C, \ldots, C)$ (M elements), $b = (1, \ldots, 1)$ (R elements), $c = d = (1, \ldots, 1)$ (N elements), and

$$a_{\bar{i}\bar{j}} := \begin{cases} w_j & \text{for } \bar{i} = 1, \ldots, m \text{ and } \bar{j} = (\bar{i}-1)n + j \quad (j \in \{1, \ldots, n\}) \\ 0 & \text{else,} \end{cases}$$

$$b_{\bar{k}\bar{j}} := \begin{cases} 1 & \text{for } \bar{k} = 1, \ldots, n \text{ and } \bar{j} = \bar{k}, n+\bar{k}, \ldots, (m-1)n+\bar{k} \\ 0 & \text{else} \end{cases}$$

as well as $x = (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{m1}, \ldots, x_{mn})$ it is readily verified that MCBCP is a representative of mixed packing covering integer problems. Note that in our case the multiplicity constraints (8) are redundant.

A special case of CIPs with generalized multiplicity constraints is obtained when B is the $N \times N$-identity matrix, i.e. when no packing constraints are existent. This problem version (i.e. (5), (6), (8), and (9)) is called CIP with multiplicity constraints and has been studied by, e.g., Dobson (1982); Kolliopoulos and Young (2001) and Kolliopoulos (2003). When no multiplicity constraints are considered at all (i.e. (5), (6), and (9)) we speak of the classical CIP (see, e.g., Srinivasan, 1999). At this point it is important to note that due to the presence of the constraints (3) we cannot formulate the MCBCP either as a CIP or as a CIP with multiplicity constraints.

With regards to algorithms for solving CIPs with generalized multiplicity constraints (and thus the MCBCP), to the best of our knowledge, merely three approximation algorithms are to be found in the literature for which, however, only analytical results are available but no computational ones. The first algorithm stems from Kolliopoulos and Young (2001) and is based on "finely" rounding a fractional optimal solution. The authors showed that for any $\varepsilon > 0$, an integral solution $\hat{x}$ of cost $\mathcal{O}(\max\{1, 1/\varepsilon^2\}(1 + (\log M)/W))$ times the optimum of the standard linear programming relaxation can be obtained in deterministic polynomial time which satisfies $A\hat{x} \geq a$ and $(B\hat{x})_r \leq \lceil (1+\varepsilon)b_r + \mathcal{O}(\min\{\varepsilon^2, 1\}\beta_r W/(\log M)) \rceil$ for all $r = 1, \ldots, R$ where β_r is the sum of coefficients at the rth row of B and W is defined as $\min\{a_i/A_{i,j} \mid A_{i,j} > 0, i = 1, \ldots, M, j = 1, \ldots, N\}$. So, obviously, the solution returned by this algorithm cannot guarantee to meet all packing constraints. Note that we have $\beta_r = n$ for all r and $W = C/w_1$ in the MCBCP.

The other two methods are bi-criteria approximation algorithms and presented in Kolliopoulos and Young (2005). According to the authors, for any $\varepsilon \in (0, 1]$, the second algorithm finds a solution $\hat{x}$ of cost $\mathcal{O}\left(1 + ln(1+\alpha)/\varepsilon^2\right)$ times the optimum, satisfying $A\hat{x} \geq a$, $B\hat{x} \leq (1+\varepsilon)b + \beta$, and $\hat{x} \leq d$ where $\beta = (\beta_1, \ldots, \beta_R)$ and α is the maximum number of covering constraints that any variable appears in. Note that we have $\alpha = 1$ in the MCBCP. Again, this algorithm cannot guarantee that its returned solution meets the packing constraints. The same applies as well to the third one which, indeed, has a better (asymptotic) cost guarantee but even violates the multiplicity constraints. Therefore, we omit any further algorithmic details and refer to Kolliopoulos and Young (2005).

In contrast to the scarce literature on CIPs with generalized multiplicity constraints there exists a considerable body of literature on CIPs/CIPs with multiplicity constraints and several specific variants thereof which, for instance, differ in restrictions on the domains of the input data. However, as our MCBCP cannot be formulated as a CIP (with multiplicity constraints), we abstain from reviewing algorithms for solving CIPs (with multiplicity constraints).

Summarizing, as can be seen from the aforementioned analytical results, the three algorithms introduced in Kolliopoulos and Young (2001) and Kolliopoulos and Young (2005) are only appropriate when B has small row sums or when no packing constraints have to be taken into account. Since neither is the case with MCBCP, application of the existing methods can yield solutions where the packing constraints are violated by a large factor. Hence, there is an obvious need for suited solution procedures.

1.3. Contribution and paper structure

It is the intention of this paper to provide the required algorithms that are capable of generating feasible and high-quality