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A new cross decomposition method for stochastic mixed-integer linear programming

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ABSTRACT

Two-stage stochastic mixed-integer linear programming (MILP) problems can arise naturally from a variety of process design and operation problems. These problems, with a scenario based formulation, lead to large-scale MILPs that are well structured. When first-stage variables are mixed-integer and second-stage variables are continuous, these MILPs can be solved efficiently by classical decomposition methods, such as Dantzig/Wolfe decomposition (DWD), Lagrangian decomposition, and Benders decomposition (BD), or a cross decomposition strategy that combines some of the classical decomposition methods. This paper proposes a new cross decomposition method, where BD and DWD are combined in a unified framework to improve the solution of scenario based two-stage stochastic MILPs. This method alternates between DWD iterations and BD iterations, where DWD restricted master problems and BD primal problems yield a sequence of upper bounds, and BD relaxed master problems yield a sequence of lower bounds. The method terminates finitely to an optimal solution or an indication of the infeasibility of the original problem. Case study of two different supply chain systems, a bioproduct supply chain and an industrial chemical supply chain, show that the proposed cross decomposition method has significant computational advantage over BD and the monolith approach, when the number of scenarios is large.

1. Introduction

Mixed-integer linear programming (MILP) paradigm has been applied to a host of problems in process systems engineering, including but not limited to problems in supply chain optim ization (Papageorgiou, 2009), oil field planning (Iyer & Grossmann, 1998), gasoline blending and scheduling (Li, Karimi, & Scrinivasan, 2010), expansion of chemical plants (Sahinidis & Grossman, 1992). In such applications, there may be parameters in the model that are not known with certainty at the decision making stage. These parameters can be customer demands, material prices, yields of the plant, etc. One way of explicitly addressing the model uncertainty is to use the following scenario-based two-stage stochastic programming formulation:

$$\begin{aligned} \min_{\substack{x_0 \\ x_1,\dots,x_s}} \ & \sum_{\omega=1}^{s} [c_{0,\omega}^T x_0 + c_\omega^T x_\omega] \\ \text{s.t.}\ & A_{0,\omega} x_0 + A_\omega x_\omega \le b_{0,\omega}, \quad \forall \omega \in \{1,\dots,s\}, \\ & x_\omega \in X_\omega, \quad \forall \omega \in \{1,\dots,s\}, \\ & x_0 \in X_0, \end{aligned} \tag{SP}$$

where $x_0 = (x_{0,i}, x_{0,c})$ includes the first-stage variables, where $x_{0,\,i}$ includes n_i integer variables and $x_{0,\,c}$ includes n_c continuous variables. Set $X_0 = \{x_0 = (x_{0,i}, x_{0,c}) : B_0 x_0 \le d_0,\ x_{0,i} \in \mathbb{Z}^{n_i},\ x_{0,c} \in \mathbb{R}^{n_c}\}$. x_ω includes the second-stage variables for scenario ω and set $X_\omega = \{x_\omega \in \mathbb{R}^{n_x} : B_\omega x_\omega \le d_\omega\}$. Parameter $b_{0,\omega} \in \mathbb{R}^m$, and other parameters in problem (SP) have conformable dimensions. Note that the second-stage variables in (SP) are all continuous.

Usually a large number of scenarios are needed to fully capture the characteristics of uncertainty; as a result, Problem (SP) becomes a large-scale MILP, for which solving the monolith (full model) using commercial solvers (such as CPLEX) may fail to return a solution or return a solution quickly enough. However, Problem (SP) exhibits a nice block structure that can be exploited for efficient solution. Fig. 1 illustrates this structure. The structure of the first group of constraints in Problem (SP) is shown by part (1) of the figure, and the structure of the last two groups is shown by part (2).

There exist two classical ideas to exploit the structure of Problem (SP). One is that, if the constraints in part (1) are dualized, Problem (SP) can then be decomposed over the scenarios and therefore it becomes a lot easier to solve. With this idea, the first group of constraints in Problem (SP) are viewed as *linking constraints*. Dantzig/Wolfe decomposition (DWD) (Dantzig & Wolfe, 1960) or column generation (Applegren, 1969; Desaulniers, Desrosiers, & Solomon, 2005) is one classical approach following

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List of Acronyms

BD	Benders decomposition
BFP	Benders feasibility problem
BPP	Benders primal problem
BMP	Benders master problem
BRMP	Benders relaxed master problem
CD	Cross decomposition
DWD	Dantzig/Wolfe decomposition
DWMP	Dantzig/Wolfe master problem
DWRMP	Dantzig/Wolfe restricted master problem
DWFRMP	Dantzig/Wolfe feasibility restricted master problem
DWPP	Dantzig/Wolfe pricing problem
DWFP	Dantzig/Wolfe feasibility pricing problem
IPP	Initial pricing problem
LD	Lagrangian decomposition
LP	Linear Programming
MILP	Mixed-integer linear program/programming
RLD	Restricted Lagrangian dual problem
SCO	Supply chain optimization
SP	Stochastic programming

this idea. In this approach, constraints in part (1) are dualized to form a pricing problem. The optimal solution of the pricing problem not only leads to a lower bound for Problem (SP), but also provides a point, or called a *column*, which is used to construct a restriction of set $\prod_{\omega=1}^{s} X_\omega$. With this restriction, a (restricted) master problem is solved, and the solution gives an upper bound for Problem (SP) and new dual multipliers for constructing the next pricing problem. Another approach following the same idea is Lagrangian decomposition/relaxation (Caroe & Schultz, 1999; Geoffrion, 1974; Guignard & Kim, 1987), where a lower bounding Lagrangian subproblem is solved at each iteration and the Lagrange multipliers for the subproblems can be generated by solving the non-smooth Lagrangian dual problem or by some heuristics. Since this idea relies on the fact that the dualization of the constraints in part (1) is not subject to a dual gap, these methods can finitely find an optimal solution of Problem (SP). If integer variables are present, the methods have to be used in a branch-and-bound framework to ensure finite termination with an optimal solution (Barnhart & Johnson, 1998; Caroe & Schultz, 1999; Frangioni, 2005), such as the branch-and-price method (Barnhart & Johnson, 1998; Desrosiers & Marco, 1989; Lubbecke & Desrosiers, 2005; Oukil, Amor, Desrosiers, & Gueddari, 2007; Vance, Barnhart, Johnson, & Nemhauser, 1994; Vanderbeck, 2000; 2006; 2011) and the branch-price-cut method (Coughlan, Lubbecke, & Schulz, 2015).

The other idea to exploit the structure is based on the fact that, if the value of x_0 is fixed, then the block column $A_{0,1}, \ldots, A_{0,s}$ in part (1) no longer links the different scenarios and therefore Problem (SP) becomes decomposable over the scenarios. With this idea, the first-stage variables are viewed as *linking variables*. Benders Decomposition (BD) (Benders, 1962) or L-shaped method (Slyke & Wets, 1969) is a classical approach following this idea. In this approach, through the principle of projection and dualization, Problem (SP) is equivalently reformulated into a master problem, which includes a large but finite number of constraints, called *cuts*. A relaxation of master problem that includes a finite subset of the cuts can be solved to yield a lower bound for Problem (SP) as well as a value for x_0. Fixing x_0 to this value yields a decomposable upper bounding problem for Problem (SP). One important advantage of BD over DWD or Lagrangian decomposition is that, finite termination with an optimal solution is guaranteed, no matter whether x_0 includes integer variables. However, when x_0 is fixed for some problems, the primal problem can have de-

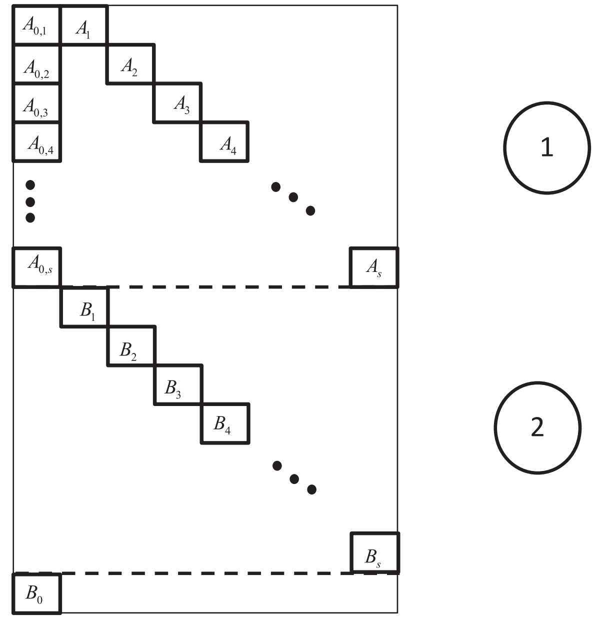

Fig. 1. The block structure of Problem (SP).

generate solutions (Contreras, Cordeau, & Laporte, 2011; Magnanti, 1981; Van Roy, 1986), resulting in redundant Bender cuts and slow convergence of the algorithm (Balas & Bergthaller, 1983; Florian, Guerin, & Bushel, 1976).

It is natural to consider synergizing the two aforementioned ideas for a unified decomposition framework that not only guarantees convergence for mixed-integer x_0, but also leads to improved convergence rate. Van Roy proposed a cross decomposition method, which solves BD and Lagrangian relaxation subproblems iteratively for MILPs with decomposable structures (Van Roy, 1983). The computational advantage of the method was demonstrated through application to capacitated facility location problems (Van Roy, 1986). Further discussions on the method, including generalization for convex nonlinear programs was done by Holmberg (1990, 1997). One important assumption of this cross decomposition method is that, the (restricted or relaxed) master problems from BD and Lagrangian relaxation are difficult to solve and should be avoided as much as possible. However, this is usually not the case for Problem (SP). Therefore, Mitra, Garcia-Herreros, and Grossmann (2014, 2016) recently proposed a different cross decomposition method, which solves subproblems from BD and Lagrangian decomposition equally frequently. They showed that their cross decomposition method was significantly faster than BD and the monolith approach for a two-stage stochastic programming formulation of a resilient supply chain with risk of facility disruption (Garcia-Herreros, Wassick, & Grossmann, 2014; Snyder & Daskin, 2005). Both Van Roy and Mitra et al. assumed that all the subproblems solved are feasible.

In this paper, we propose a new cross decomposition method which has two major differences from the cross decomposition methods in the literature. First, we combine BD and DWD instead of BD and Lagrangian decomposition in the method. Second, we solve the subproblems in a different order. In addition, we include in the method a mechanism so that the algorithm will not be stuck with infeasible subproblems. In order to simplify our discussion, we rewrite Problem (SP) into the following form:

$$\begin{aligned} \min_{x_0,x}\ & c_0^T x_0 + c^T x \\ \text{s.t.}\ & A_0 x_0 + A x \le b_0, \\ & x \in X, \\ & x_0 \in X_0, \end{aligned} \tag{P}$$