Operations Research Perspectives 4 (2017) 113–117

Contents lists available at ScienceDirect

Operations Research Perspectives

journal homepage: www.elsevier.com/locate/orp

A note on the optimal pricing strategy in the discrete-time Geo/Geo/1 queuing system with sojourn time-dependent reward

Doo Ho Lee

Department of Industrial and Management Engineering, Kangwon National University, 346 Joongang-ro, Samecheok-si, Gangwon-do 29513, Republic of Korea

ARTICLE INFO

Article history:
Received 14 October 2016
Revised 5 July 2017
Accepted 25 August 2017
Available online 1 September 2017

Keywords:
Optimal pricing
Equilibrium behavior
Geo/Geo/1 queue
Sojourn time-dependent reward

ABSTRACT

This work studies the optimal pricing strategy in a discrete-time Geo/Geo/1 queuing system under the sojourn time-dependent reward. We consider two types of pricing schemes. The first one is called the ex-post payment scheme where the server charges a price that is proportional to the time a customer spends in the system, and the second one is called ex-ante payment scheme where the server charges a flat price for all services. In each pricing scheme, a departing customer receives the reward that is inversely proportional to his/her sojourn time. The server should make the optimal pricing decisions in order to maximize its expected profits per time unit in each pricing scheme. This work also investigates customer's equilibrium joining or balking behavior under server's optimal pricing strategy. Numerical experiments are also conducted to validate our analysis.

1. Introduction

Researches about the economic analysis of queuing systems can go back at least to the pioneering work of Naor [1] who investigated customer's equilibrium and socially optimal strategies in the observable M/M/1 queuing system with the concise reward-cost structure. Naor's [1] work has been extended by many authors [2–15]. Recently, several studies have extended this topic to the discrete-time queuing system. Ma et al [16] analyzed customers' equilibrium behaviors in the discrete-time Geo/Geo/1 queue with multiple vacations, and Wang et al [17] considered the discrete-time Geo/Geo/1 queue with the single working vacation. Yang et al [18] also studied customers' balking strategies in the discrete-time Geo/Geo/1 with server breakdowns and repairs. However, to the best of our knowledge, there are relatively few works on pricing problems in the discrete-time queue. Although Ma and Liu [19] insisted that the pricing problems in the discrete-time Geo/Geo/1 queue are analyzed, in fact, the results of Ma and Liu [19] are about those in the continuous-time M/M/1 queue.

In this work, we analyze the pricing strategies and customers' equilibrium joining/balking behaviors in the discrete-time Geo/Geo/1 queuing system under two types of pricing structures. In the first structure, the server charges a price that is proportional to the sojourn time (waiting time plus service time), called the ex-post payment (EPP) scheme. In the second structure, however, the server charges a flat price for all the services, which means the server implements the ex-ante payment (EAP) scheme. In each pricing structure, the server should make the optimal pricing decisions in order to maximize its expected profits per time unit. For more information on the EPP and EAP schemes, refer to [19–22].

By the way, most of previous studies assume that the reward that a departing customer receives after being served is set to a constant value. In the service system, however, the service quality is adversely affected by the system congestion. Demand growth in the queuing system can have two diametrically opposite effects on the server's pricing strategy: one is a price hike, as is common in economics; the other is an increase in congestion, which deteriorates the service quality and thus implies a lower price. A key performance measure of the system congestion is customer's sojourn time. It is a matter of course that rewards, which can be expressed by customer satisfaction, are influenced by the sojourn time. We frequently observe that the longer the sojourn time, the lower the customer satisfaction and the lower system operational profit. For example, if the line is too long, customers can give up and go to another service system. Long queues have a negative impact on customer service satisfaction, which causes service abandonment. This is an opportunity cost for the system, which has a negative impact on profitability. For this reason, we assume that the reward is adversely proportional to customer's sojourn time.

E-mail address: enjdhlee@kangwon.ac.kr

http://dx.doi.org/10.1016/j.orp.2017.08.001

The remainder of this paper is organized as follows. In Section 2, the mathematical model under consideration is described. In Section 3, we analyze customer's equilibrium joining/balking behavior and server's profit maximization strategy under the EPP and EAP schemes, respectively. Section 4 deals with numerical experiments where we investigate trends of the optimal prices of the EPP and EAP schemes according to various input values.

2. Model description

This work considers a queuing system with following features. The arrival process of potential customers is the Bernoulli process with a probability of $p\,(0<p<1)$. Whenever each potential customer arrives at the system, they decide to join or balk depending on the specific joining probability, denoted by $q\ (0\le q\le 1)$. It is well known that the decomposition property holds in the Bernoulli process; therefore, the effective arrival process follows the Bernoulli process with a probability of pq. Service times are independent and identically distributed random variables (RVs) following geometric distribution with a probability of $\mu\ (0<\mu<1)$. For analytical simplicity, we assume followings: i) arrival and service processes are mutually independent; ii) services are provided on a first-in-first-out basis; iii) the decision to join or balk is irrevocable; iv) the stable system should satisfy $pq<\mu$.

Let π_n be the stationary queue length distribution. According to Takagi [23], π_n is given by

$$\pi_0 = 1-\frac{pq}{\mu} \quad \text{and} \quad \pi_n = \frac{\mu-pq}{\mu(1-\mu)}\left(\frac{pq(1-\mu)}{\mu(1-pq)}\right)^n, \quad n\ge 1. \tag{1}$$

Let L and ω respectively denote the expected queue length and the expected sojourn time. L is then given by $L=\sum_{n=1}^{\infty} n\pi_n = pq(1-pq)/(\mu-pq)$. By Little's formula, $\omega=(1-pq)/(\mu-pq)$. Due to the fact that $\mu<1$, we have $\omega>1$.

Let R denote the reward that each customer expect to receive before being served. As mentioned in Section 1, the sojourn time have a negative effect on the reward. After the completion of a service, therefore, each customer is assumed to receive a reward R/ω $(R>0)$. There exists a waiting cost C $(C>0)$ per time unit when the customer stays in the system. To make the model nontrivial, we assume the condition that $R\mu>C/\mu$, which ensures that the expected reward exceeds the expected cost for the customer finding the empty system.

3. Pricing analysis

3.1. EPP scheme

We first investigate the EPP scheme, where the server charges a price that is proportional to customer's sojourn time. Let K_t $(K_t\ge 0)$ and P_t denote the price charged by the server and the expected server profit per time unit, respectively. Then, the expected utility U_t has the following relation: $U_t=R/\omega-K_t\omega-C\omega$. If U_t equals zero at customer's equilibrium, we have $R=(K_t+C)\omega^2$. At customer's equilibrium, therefore, the price can be expressed in terms of customer's joining probability, denoted by q_t

$$K_t = \frac{R(\mu-pq_t)^2 - C(1-pq_t)^2}{(1-pq_t)^2}. \tag{2}$$

Substituting (2) into $P_t=pq_tK_t\omega$, we have

$$P_t = \frac{pq_t\left(R(\mu-pq_t)^2 - C(1-pq_t)^2\right)}{(\mu-pq_t)(1-pq_t)}. \tag{3}$$

We now establish the following non-linear programming (NLP) problem to maximize P_t with respect to q_t:

$$\begin{aligned} &\max_{q_t} P_t = \tfrac{pq_t\left(R(\mu-pq_t)^2-C(1-pq_t)^2\right)}{(\mu-pq_t)(1-pq_t)} \\ &\text{st} \\ &0\le q_t\le 1, \\ &pq_t<\mu. \end{aligned} \tag{4}$$

In (4), we want to maximize the expected server profit per time unit. The first constraint implies that customer's joining probability should be bounded between 0 and 1, and the second one guarantees the system to be stable. We now introduce the following lemma:

Lemma 1. *The optimization problem in (4) is a convex maximization problem (CMP).*

Proof. According to the definition of the convex maximization [24], if the objective function can be proved concave in the feasible region and the set of constraints can be proved convex, the maximization problem is a CMP.

The constraint in (4) are all real-valued linear functions; therefore, the set of constraints is convex. The second derivative of the objective function in (4) can be expressed as

$$\frac{\partial^2 P_t}{\partial q_t^2} = -\frac{2p^2(1-\mu)\left[R(\mu-pq_t)^3 + C\mu(1-pq_t)^3\right]}{(1-pq_t)^3(\mu-pq_t)^3}. \tag{5}$$

From the constraints in (4), $(\mu-pq_t)^3>0$ and $(1-pq_t)^3>0$. Thus, $\partial^2P_t/\partial q_t^2<0$ in the feasible region. Hence, we conclude that the objective function P_t should be strictly concave and (4) is a CMP. □

Since the optimization problem in (4) is a CMP, the Lagrange multiplier method can be used to find the optimal solution. Observing (4), we can find that all constraints are inequality constraints. Thus, Karush-Kuhn-Tucker conditions can be used to generalize the method of Lagrange multiplier. However, the explicit form of the optimal q_t, denoted by q_t^*, is too long and complicated. We instead introduce another way of obtaining the value of q_t^*. If the objective function is strictly concave in the feasible region, the unique local optimal solution obtained by using the Newton method becomes the unique global optimal solution. For the detailed Newton method, refer to Boyd and Vandenberghe [24]. Then, the optimal price K_t^* is expressed as

$$K_t^* = \frac{R(\mu-pq_t^*)^2 - C(1-pq_t^*)^2}{(1-pq_t^*)^2}. \tag{6}$$

Based on the above analysis, we could give the following theorem:

Theorem 1. *Under the EPP scheme, if $0\le q_t\le 1$, there exists a unique equilibrium where customers join the queue with a probability of pq_t^*.*

Proof. Let $\omega(q_t)=(1-pq_t)/(\mu-pq_t)$. As mentioned, the expected utility has the following relation: $U_t=R/\omega(q_t)-(K_t+C)\omega(q_t)$. We distinguish three cases:

Case 1: $R/\omega(0)\le(K_t+C)\omega(0)$. In this case, even if no other customer joins, the expected benefit of a customer who joins is non-positive. Therefore, the strategy of joining with probabilities $q_t^*=0$ is an equilibrium strategy and no other equilibrium is possible. Moreover, in this case, not joining is a dominant strategy.

Case 2: $R/\omega(1)\ge(K_t+C)\omega(1)$. In this case, even if all potential customers join, they all enjoy a non-negative benefit. Therefore, the strategy of joining with probability $q_t^*=1$ is an equilibrium strategy and no other equilibrium is possible. Moreover, in this case, joining is a dominant strategy.