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# Hybrid approach for solving systems of nonlinear equations using chaos optimization and quasi-Newton method

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**Abstract**

Solving systems of nonlinear equations is one of the most difficult numerical computation problems. The convergences of the classical solvers such as Newton-type methods are highly sensitive to the initial guess of the solution. However, it is very difficult to select good initial solutions for most systems of nonlinear equations. By including the global search capabilities of chaos optimization and the high local convergence rate of quasi-Newton method, a hybrid approach for solving systems of nonlinear equations is proposed. Three systems of nonlinear equations including the ‘‘Combustion of Propane’’ problem are used to test our proposed approach. The results show that the hybrid approach has a high success rate and a quick convergence rate. Besides, the hybrid approach guarantees the location of solution with physical meaning, whereas the quasi-Newton method alone cannot achieve this.


*Keywords:* Systems of nonlinear equations; Chaos optimization; Quasi-Newton method; Hybrid approach

## 1. Introduction

Solving systems of nonlinear equations is perhaps one of the most difficult problems in all of numerical computation and in a spectrum of engineering applications such as numerical weather forecasting, electric power generation and distribution, computational biochemistry, and trajectory/path-planning applications. Tremendous efforts have been made to solve systems of nonlinear equations and progress along this line now include a number of constructive theories and algorithms related to systems of nonlinear equations [1–4]. However there still exist some obstacles in solving systems of nonlinear equations. As is well known, Newton-type methods are the most widely used algorithms for solving systems of nonlinear equations, but their convergence and performance characteristics are highly sensitive to the initial guess of the solution supplied to the methods.

Chaos, apparently disordered behavior that is nonetheless deterministic, is a universal phenomenon that occurs in many systems in all areas of science [5]. The chaos optimization algorithm (COA) is based on ergodicity, stochastic properties and ‘‘regularity’’ of the chaos [6]. The COA is not like some stochastic optimization algorithms that escape from local minima by accepting some bad solutions according to certain probability. It searches on the regularity of chaotic motion. Recently, several chaos optimization algorithms have been developed to deal with optimization problems. Li and Jiang [6] developed a chaos search method of ‘‘carrier wave’’ to optimize problems. A mutative scale chaos optimization algorithm was proposed based on continually reducing the searching space of variables by Zhang et al. [7], and this method enhanced the searching precision. Liu and Hou [8] proposed a chaos optimization algorithm using a weighted gradient direction for nonlinear programming problem. In their studies, chaos optimization algorithms were confined to optimize some classical functional problems. Although these COAs can locate the global solution, it was at the cost of long computation time. The main goal of this paper is to extend COA to solve systems of nonlinear equations. A hybrid approach combining chaos search and Newton-type methods to solving systems of nonlinear equations is presented.

The paper is organized as follows: Section 2 is the problem description. Section 3 makes a simple survey on related works. Section 4 outlines the chaos optimization algorithm. The hybrid

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approach combining the COA and quasi-Newton method is described in Section 4. Numerical results of two nonsmooth systems of nonlinear equations are listed in Section 5 and the application of the proposed algorithm to a practical problem is provide in Section 6. The major results of this study are summarized in Section 7.

## 2. Problem description

Assume that a system of nonlinear equations is soluble and its solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, the general formulation of a system of nonlinear equations is

$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) = 0 \\ f_2(x_1, x_2, \ldots, x_n) = 0 \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) = 0 \end{cases} \quad (1)$$

To solve the system of nonlinear equation is equivalent to minimizing a master function described as follows:

$$\begin{cases} \text{Find}: & \mathbf{x} = (x_1, x_2, \cdots, x_n)^T, \mathbf{x} \in \Phi \\ \text{Min}: & F(\mathbf{x}) = \sum_{i=1}^{n} f_i^2(\mathbf{x}) \end{cases} \quad (2)$$

where $\Phi$ is the solution space. This master function is positive definite and has a global minimum at each of the roots. When the minimization of $F(\mathbf{x})$ is 0, the corresponding $\mathbf{x}$ is the exact solution.

Newton's method is the most widely applied iterative method for solving systems of nonlinear equations. Newton's method is driven by derivative information and derived from Taylor's series analysis. In the solution of a system of nonlinear equations, the method employs a linear model. The iterative format of Newton's method for solving a system of equations can be written as follows:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^k - J(\mathbf{x}^{(k)})^{-1} f(\mathbf{x}^{(k)}) \quad (3)$$

where $J(\mathbf{x})$ is the Jacobian matrix

$$J(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x})}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n(\mathbf{x})}{\partial x_1} & \frac{\partial f_n(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_n(\mathbf{x})}{\partial x_n} \end{bmatrix} \quad (4)$$

Because Newton's method has several disadvantages such as high time cost for Jacobian matrix, inability to ill-conditioned matrix, etc., a lot of revised Newton-type methods have been introduced, among of which the quasi-Newton method is the representative one [1–3]. Compared with Newton's method, the quasi-Newton method replaces derivative computation with direct function computation.

Once the problem of solving systems of nonlinear equations is changed into an optimization problem described as Eq. (2), the classical nonlinear optimization algorithms such as conjugate gradient, simplex method, Powell and BFGS can all be applied to solve a system of nonlinear equations. However, the majority of these methods are not always effective in the minimization of the master function stemming from the problem of solving a system of nonlinear equations [12].

## 3. Related work

The Newton's method and its variations including inexact Newton method, quasi-Newton method are widely used for solving a system of nonlinear equations because of their computational efficiency [1–3,14]. These methods can achieve super-linear convergence if the initial starting point is within a certain neighborhood of the solution. However, this type of method fails if the initial guess is not sufficiently close to the solution, or if singular points are encountered. In other words, these methods are locally convergent and cannot provide guarantees for obtaining solutions. Homotopy methods have been developed to overcome the disadvantages of Newton-type methods [3,14]. It is generally true that Homotopy methods are more reliable than Newton-type methods. However, the extra robustness comes at a price, since Homotopy methods typically require significantly more function and derivative evaluations and linear algebra operations than the Newton-type methods. Besides, they either do not guarantee to locate a solution even for a fairly simple problem [3]. In a word, these existing algorithms for solving systems of nonlinear equations have more or less disadvantages. The high efficient, reliable algorithms for solving systems of nonlinear equations need further investigation.

The advanced soft computation techniques including evolutionary algorithms, simulated annealing, swarm algorithm, and neural network algorithms, etc, have been proved to be superior to the classical algorithms in solving the large-scale and difficult problems. Therefore, it is desirable to solve systems of nonlinear equations using these soft computation techniques.

Zhao and Chen [9] proposed a neural network model for solving systems of nonlinear equations, which is strictly proved to be stable and useful to solve not only systems of nonlinear equations, but also nonlinear multivariable equations and systems of linear equations. Nguyen [10] established a general neural network architecture, which can achieve ultrahigh-speed computation for solving large systems of nonlinear equation.

Hu et al. [11] established a common solution model for all kinds of algebraic equations group based on a genetic algorithm (GA). A hybrid scheme for solving systems of nonlinear equations is presented by Karr et al. [12], in which a GA is used to locate efficient initial guesses, which are then supplied to a Newton's method. Luo et al. [13] proposed a different hybrid scheme combining GA and the class algorithms (Powell algorithm and Newton's method) for solving systems of nonlinear equations.

It has been shown that the COA is much powerful than some stochastic optimization algorithms such as simulated annealing, chemotaxis algorithm by the application to complex functional optimization [6]. The applications of the COAs to mesh optimization [15], structure design [16] and fuzzy control rules design [17] show that COAs are an effective method to