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Adaptive finite element method for parabolic equations with Dirac measure

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Abstract

In this paper we study the adaptive finite element method for parabolic equations with Dirac measure. Two kinds of problems with separate measure data in time and measure data in space are considered. It is well known that the solutions of such kind of problems may exhibit lower regularity due to the existence of the Dirac measure, and thus fit to adaptive FEM for space discretization and variable time steps for time discretization. For both cases we use piecewise linear and continuous finite elements for the space discretization and backward Euler scheme, or equivalently piecewise constant discontinuous Galerkin method, for the time discretization, the a posteriori error estimates based on energy and L^2 norms for the fully discrete problems are then derived to guide the adaptive procedure. Numerical results are provided at the end of the paper to support our theoretical findings.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3 be an open bounded convex polygonal or polyhedral domain with Lipschitz boundary $\Gamma = \partial\Omega$ and $T > 0$ be a real number. The purpose of this paper is to consider the finite element approximations of the following parabolic equations with measure data in space

$$\begin{cases} \partial_t y + \mathcal{A}y = g(x,t)\delta_{\gamma(t)} & \text{in } \Omega_T, \\ y = 0 & \text{on } \Gamma_T, \\ y(\cdot,0) = y_0 & \text{in } \Omega \end{cases} \tag{1.1}$$

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and parabolic equations with measure data in time

$$\begin{cases} \partial_t y + \mathcal{A}y = g(x,t)\delta_{t_0} & \text{in } \Omega_T, \\ y = 0 & \text{on } \Gamma_T, \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\Omega_T = \Omega \times (0,T)$ and $\Gamma_T = \partial\Omega \times (0,T)$, $\partial_t y = \frac{\partial y}{\partial t}$, the operator $\mathcal{A}$ is a second order elliptic partial differential operator, $y_0 \in L^2(\Omega)$ is the given initial condition, g is a given function such that $g \in L^2(0,T;\mathcal{C}(\overline{\Omega}))$ for (1.1) and $g \in \mathcal{C}([0,T];L^2(\Omega))$ for (1.2).

In Eq. (1.1), we assume that $\gamma(t)$ is a lower dimensional time-continuous manifold which is strictly contained in Ω for all $t \in [0,T]$ (see [1,2] for similar definition). $\delta_{\gamma(t)}$ denotes the Dirac measure in space on $\gamma(t)$. We note that $\gamma(t)$ can be a point, a curve or even a surface if $d = 3$, it can be static and independent of time t or evolves in the time horizon. For simplicity we assume that $\gamma(t)$ is a Lipschitz-continuous m-dimensional manifold in Ω with $0 \leq m \leq d-1$ for all $t \in [0,T]$, and the distance between $\gamma(t)$ and $\partial\Omega$ is positive for all $t \in [0,T]$. The m-dimensional Hausdorff measure of $\gamma(t) \subset \Omega$ in $\mathbb{R}^d$ is finite for all $t \in [0,T]$. When $m = 0$, $\gamma(t)$ will reduce to a single point or a finite number of points for each $t \in [0,T]$; when $m = 1$, $\gamma(t)$ is a $\mathcal{C}^2$-curve s.t. $\gamma(t) \subset \partial D$ for some d-dimensional $\mathcal{C}^2$-domain $D \subset\subset \Omega$ for each $t \in [0,T]$; when $m = 2$, $\gamma(t)$ is a $\mathcal{C}^2$-surface s.t. $\gamma(t) \subset \partial D$ for some 3-dimensional $\mathcal{C}^2$-domain $D \subset\subset \Omega$ for each $t \in [0,T]$. In Eq. (1.2) we assume that δ_{t_0} denotes the Dirac measure in time on given point $t_0 \in (0,T)$.

The problems of form (1.1) with measure data in space can be used to model the potential of an electric field with an electric charge distribution. This kind of problems also arise in other different applications, for instance, modeling of acoustic monopoles, transport equations for effluent discharge in aquatic media, and so on. Also, there are some applications in inverse problems where one attempts to identify the moving pointwise sources in the heat and convection–diffusion equations, see for example [3]. One of the most import applications of parabolic equations with measure data appears in optimal control theory. For instance, problems of form (1.1) with measure data in space can serve as the state equation of some parabolic optimal control problems with pointwise control (see [4,5] and [6] for more details) and sparse controls (see [7]), or can be used to model air or water pollution control problems (see [8]). Moreover, parabolic optimal control problems with controls acting on a lower dimensional manifold also involve such kind of parabolic equations with Dirac measure in space where the measure data may evolve in the time horizon, we refer to [1] and [2] for more details.

On the other hand, parabolic equations of form (1.2) with measure data in time also appear in the optimality conditions of some optimal control problems with state constraints, for example, pointwise state constraints in time, and serve as the so-called adjoint state equation, we refer to [9] and [10] for more details on this kind of optimal control problems.

There have already appeared some contributions to the theoretical and numerical analysis for partial differential equations with measure data. Boccardo and Gallouët studied the existence of solutions for quasi-linear elliptic and parabolic equations involving measure data in [11,35], Casas studied in [9] the linear parabolic problems with measure data and improved the results of [11] by exploiting the linearity of the equation. The finite element method for elliptic equation with Dirac measure data has been extensively studied (see, e.g., [12] and the references cited therein). Casas gave an optimal error estimate of order $O(h^{2-\frac{d}{2}})$ in [12], where h is the mesh size of space triangulation and d is the dimension of Ω. Based on the similar duality argument Gong derived a priori error estimates for finite element approximations of parabolic equations with measure data in [13].

It is well known that the solutions of PDEs with measure data exhibit low regularity, thus the well developed adaptive finite element method fits to this kind of problems for the sake of accuracy enhancement. To this end, the a posteriori error estimators should be constructed to guide the adaptive procedure, which is the main purpose of this paper. We refer to [14] for an excellent review for a posteriori error estimates of different types. As for the a posteriori error estimates and adaptive finite element methods for parabolic equations, we mention the earlier work of Eriksson and Johnson in [15] and [16], and followed by Picasso in [17], Chen and Jia [18] and Kreuzer et al. [19]. We do not aware of much work on the a posteriori error estimates for PDEs with measure data, among them we should mention the work of Araya et al. in [20], where a posteriori error estimates for elliptic problems with Dirac delta source terms are derived. In this work we intend to derive the a posteriori error estimators for parabolic equations with measure data in both space and time. To the best of our knowledge this is the first contribution on this subject in the literature. We emphasize that our problems are different from the case studied in [18,19] and [17], where the right hand sides