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C^0-discontinuous Galerkin methods for a wind-driven ocean circulation model: Two-grid algorithm

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Abstract

This paper presents a nonconforming finite element method for a streamfunction formulation of the stationary quasi-geostrophic equations, which describe the large scale wind-driven ocean circulation. The streamfunction formulation is a fourth order nonlinear PDE and the nonconforming method is based on C^0-elements instead of C^1-elements. Existence and uniqueness of the approximation are proved and optimal error estimates in several norms of interest are demonstrated under a small data assumption. Two-grid algorithms based on Picard and Newton type linearizations are then presented to efficiently resolve nonlinearities and computational results are given to demonstrate the efficiency of the algorithm. The Mediterranean sea example is tested with real world coastline data, which illustrates the effectiveness of the two-grid approach in the wind-driven ocean circulation simulation.

Keywords: Nitsche's method; Nonconforming finite-element method; Fourth-order partial-differential equation; Geophysical fluid dynamics

1. Introduction

The oceans are one of the most important elements in the climate system. They play a critical role in storing heat and their large scale currents influence the climate by transporting heat around the globe [1,2]. The large scale wind-driven ocean currents are characterized by the wind forcing, the stratification, and the effects of rotation. Annual mean wind patterns are westward near the equator and eastward at the midlatitudes. These wind patterns and the Coriolis force drive strong western boundary currents such as the Gulf stream in the North Atlantic and the Kuroshio in the North Pacific. These western boundary currents are much narrower and faster than eastern boundary currents such as the Canary current in the North Atlantic and the California current in the North Pacific.

The quasi-geostrophic equations (QGE) are a popular mathematical model of the large scale wind-driven ocean circulation [3–8]. As the QGE gets more attention, many researchers have proposed and developed various numerical methods to solve the equations. The finite difference method [9,10] and finite volume method [11,12] have been

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mostly studied. Recently, researchers began to apply finite element approach because this approach helps ease boundary treatment and refinement to achieve a high resolution in regions of interest [13–17].

In this paper, we consider the streamfunction formulation of the one-layer stationary QGE:

$$Re^{-1}\Delta^2\psi + J(\psi, \Delta\psi) - Ro^{-1}\frac{\partial \psi}{\partial x} = Ro^{-1}F \qquad \text{in } \Omega, \tag{1.1}$$

with boundary conditions

$$\psi = 0 \quad \text{and} \quad \nabla\psi \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$ is a plane domain with boundary Γ, ψ is the velocity streamfunction, F is the forcing term, $\mathbf{n}$ is the outward unit normal vector on Γ, and

$$J(\psi, \Delta\psi) := \frac{\partial \psi}{\partial x}\frac{\partial \Delta\psi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \Delta\psi}{\partial x}$$

is the Jacobian operator. The Reynolds number Re and Rossby number Ro are defined as

$$Re = \frac{UL}{A} \quad \text{and} \quad Ro = \frac{U}{\beta L^2},$$

where A is the eddy viscosity parametrization, U is a characteristic velocity, L is the width of computational domain, and β is the coefficient multiplying the y-coordinate in the β-plane approximation (see [3]). For large scale oceanic flows, Re is large and Ro is small, and thus, the stationary QGE are dominated by convective terms with large forcing.

The stationary QGE model (1.1) is a fourth order PDE and C^1-elements are a natural choice for finite element approximations. The C^1-conforming finite element method is presented in [15]. There, optimal convergence estimates are derived and the unique solvability is investigated. The method is flexible to treat boundary conditions. However, it requires at least 5th order polynomials. For the computational efficiency, cubic B-spline based C^2 finite element method is introduced in [16]; boundary treatments remain as an issue. On the other hand, the streamfunction–vorticity formulation of the QGE are discussed in [3,5,18,19]. Since this formulation consists of second order PDEs, conforming C^0-elements can be used for the approximation. Indeed, in [13], the author proposed the finite element method with C^0-element for the streamfunction–vorticity formulation, and suboptimal error estimates were obtained.

Recently, the C^0 discontinuous Galerkin (C^0-DG) method has been proposed for the stationary QGE in [17]. The method weakly enforces continuity of the derivatives across interfaces via Nitsche's method [20] and enables quadratic approximations as the lowest order approximation. This approach is relevant to a consistent C^0-interior penalty method introduced by Engel et al. [21] and developed by Brenner et al. [22–24], Wells and Dung [25] and Hansbo et al. [26]. Since the QGE model problem is nonlinear, it is inevitable to use an appropriate iterative method in the process of solving the problem. One of the method to reduce computational cost is the two-grid algorithm introduced by Xu [27]. This algorithm helps obtain the numerical solution for the nonlinear PDEs efficiently by using two finite element subspaces, namely, S^H and S^h (with mesh size $h \ll H$). It is worth noting that a very coarse grid space is usually sufficient to handle the nonlinearity. For more details on two-grid methods, see, for example, [28] and references cited therein.

In this paper, we further study the C^0-DG method for the stationary QGE. In the previous work [17], the C^0-DG formulations and their numerical results were presented for both the linear Stommel–Munk model and nonlinear QGE model. Coercivity and consistency of the method were proved as well for both models. However, their error analysis was limited to the linear Stommel–Munk model only. Compared with the earlier effort [17], several contributions are made in this work: First, we provide theoretical results for the nonlinear QGE model. More precisely, well-posedness of the nonlinear discrete problem is proved based on Brouwer's fixed point argument [29–31] under a small data assumption, and the error estimates are derived in several norms of interest. Next, two types of two-grid algorithms for the C^0-DG method are presented and investigated. One is based on the usual Newton type linearization and the other utilizes a simple Picard iteration. Several numerical experiments reveal that both two-grid algorithms outperform the one-grid method of [17]. Also, performance of two-grid algorithms changes with the order of approximation (P_2 and P_3) as well as the existence of a strong layer. Finally, two-grid algorithms are tested on unstructured meshes. The Mediterranean sea example is considered with real world coastline data, which illustrates the effectiveness of the two-grid approach in the wind-driven ocean circulation simulation.

The remainder of the paper is organized as follows. In the next section, we introduce preliminaries for nonconforming finite element method. In Section 3, we present our weak formulation for the model problem,