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# Dispersion-minimizing quadrature rules for $C^1$ quadratic isogeometric analysis

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**Abstract**

We develop quadrature rules for the isogeometric analysis of wave propagation and structural vibrations that minimize the discrete dispersion error of the approximation. The rules are optimal in the sense that they only require two quadrature points per element to minimize the dispersion error (Bartoň et al., 2017 [1]), and they are equivalent to the optimized blending rules we recently described. Our approach further simplifies the numerical integration: instead of blending two three-point standard quadrature rules, we construct directly a single two-point quadrature rule that reduces the dispersion error to the same order for uniform meshes with periodic boundary conditions. Also, we present a 2.5-point rule for both uniform and non-uniform meshes with arbitrary boundary conditions. Consequently, we reduce the computational cost by using the proposed quadrature rules. Various numerical examples demonstrate the performance of these quadrature rules.


*Keywords:* Isogeometric analysis; Quadrature rule; Dispersion analysis; Spectrum analysis

## 1. Introduction

Quadrature rules play an important role in the implementation of various numerical methods for solving partial differential equations. Fewer quadrature points result in a lower computational cost, however, the reduction of the quadrature points should not reduce the quality of the approximation. The design of efficient quadrature rules for isogeometric analysis (see Hughes et al. [2–5]) is of interest as the continuity properties of the spline basis functions may require fewer quadrature points. The quadrature rules should preserve the optimal convergence of the numerical approximation to the exact solution. Traditionally, Gauss rules for discontinuous polynomial spaces are used, however, these choices are far from being optimal in general [6].

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The construction of efficient quadrature rules for isogeometric analysis was initially considered by Hughes et al. in [7] in 2010. Taking advantage of the smoothness of the basis functions across element boundaries, a half-point rule that is independent of the polynomial order of the basis functions was developed. The new rule has advantages when compared to the traditional ones. The rule is optimal as it exactly integrates the spline basis functions with the minimum number of quadrature sampling points. The rule is designed for uniform univariate splines and is Gaussian, that is, optimal in the sense of the minimum number of quadrature points. However, the rule is exact only for infinite domains or for the spline spaces that have a special structure at the boundaries of finite domains. To make the rule exact for a general spline space over finite domains, additional quadrature points are introduced at the boundary elements, resulting in nearly-optimal quadrature rules [5]. These non-Gaussian rules come as solutions of non-linear, possibly ill-conditioned, systems and possess both positive and negative weights.

Other works in this direction are reported in [8–15]. Optimal and reduced quadrature rules for tensor product and hierarchically refined splines for isogeometric analysis were developed in [9]. Gaussian rules for spline spaces of various degrees and continuities were derived in [11,12]. Using the homotopy continuation argument [10], Gaussian rules can be derived by continuously modifying the spline space (knot vector) and by tracing numerically the rule, which is given by solving a certain algebraic system. These rules guarantee exactness of the integration up to machine precision, and the property of being Gaussian also directly implies that all weights are positive [16]. Recently, Calabró and his collaborators [14] changed the paradigm of the assembly of Galerkin matrices from the traditional element-wise to a row-wise concept. For each row of the mass and stiffness matrices, they compute its own weighted quadrature by solving a linear system. This brings significant computational savings as the total cost does not depend exponentially on the polynomial degree, but requires only two quadrature points per element, regardless the degree. In [15], the authors proposed a new reduced quadrature rule for isogeometric analysis and these quadrature rules were derived based on the idea of variational collocation and Cauchy's first mean value theorem of integral calculus. The number of quadrature points are reduced significantly and hence gain computational efficiency.

The study of dispersion error minimization for isogeometric analysis was initially studied numerically in Puzyrev et al. [17] and analytically in Calo et al. [18]. For general dispersion analysis of isogeometric discretizations, we refer the readers to [19,20] and the references therein. Particularly, in Hughes et al. [19], a duality principle between the dispersion analysis and the spectral analysis was established and the analysis unified.

The study of dispersion analysis of the finite element method has a rich literature; see for example Thomson and Pinsky [21,22], Ihlenburg and Babuska [23], Ainsworth [24–26], and others [27–29]. Thomson and Pinsky studied the dispersive effects of the finite element methods with Legendre, spectral, and Fourier local approximation basis for the Helmholtz equation in [21]. They found that the choice of basis functions had a negligible effect on the dispersion errors. This is due to the low continuity ($C^0$ continuity) of the basis functions. Hughes et al. [19] showed that the dispersion error of the isogeometric analysis with high continuity (up to $C^{p-1}$ for $p$th order basis function) on the basis functions is smaller than that of the lower continuity finite element counterparts.

The $2p$-optimal convergence rate of the dispersion error for the $p$th order standard finite elements was established in [24]. In 2009, Ainsworth and Wajid [25] extended this analysis to arbitrary spectral element methods. Based on Marfurt's conjecture [30] that the most promising and efficient method for computing wave propagation is to blend the finite element method with the spectral element method with appropriate weights, Ainsworth and Wajid beautifully established the optimal blending of these two methods in [26]. A superconvergence (order $2p + 2$ for $p$th order polynomial approximation) result was obtained for arbitrary order of polynomial approximation, which includes the fourth order superconvergence result obtained by a modified integration rule for linear finite elements in [31].

To the best of our knowledge, this is the first paper studying the design of optimal quadrature rules which minimize the dispersion errors of the isogeometric analysis for the wave propagation and structural vibration problems. The dispersion error-minimizing quadratures, that combine Gauss–Legendre and Gauss–Lobatto rules proposed in [17,18], are not efficient as the two traditional quadrature rules are used for each integration evaluation. Herein, we design quadrature rules that minimize the dispersion error and minimize the number of quadrature points. A rule that has minimal number of evaluation points per element (two in the case of a uniform mesh with periodic boundary conditions [1]) is the solution of a non-linear system of algebraic equations which, due to the low polynomial degree, admits a closed form formula. We also design a quadrature rule that minimizes dispersion for the larger $C^0$ quadratic space, which leads to a quadrature rule that uses 2.5 points per element as it exactly integrates discontinuous cubic functions on the mesh. This rule is effective for non-uniform meshes and arbitrary boundary conditions.