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Higher-order meshing of implicit geometries, Part II: Approximations on manifolds

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Highlights

- Automatic, higher-order mesh generation of curved surfaces from level-set data.
- Bounded surface is defined by several level-set functions on a higher-order background mesh.
- The background mesh is manipulated to avoid degenerated elements.
- Mesh generation from individual elements is based on connectivity data of the background mesh.
- Optimal convergence rates on the automatically reconstructed meshes is achieved using standard surface FEM.

Abstract

A new concept for the higher-order accurate approximation of partial differential equations on manifolds is proposed where a surface mesh composed by higher-order elements is automatically generated based on level-set data. Thereby, it enables a completely automatic workflow from the geometric description to the numerical analysis without any user-intervention. A master level-set function defines the shape of the manifold through its zero-isosurface which is then restricted to a finite domain by additional level-set functions. It is ensured that the surface elements are sufficiently continuous and shape regular which is achieved by manipulating the background mesh. The numerical results show that optimal convergence rates are obtained with a moderate increase in the condition number compared to handcrafted surface meshes.

Keywords: Higher-order FEM; Manifold; Surface PDEs; Level-set method

1. Introduction

Many challenging applications in engineering and natural sciences are characterized by physical phenomena taking place on curved surfaces in the three-dimensional space. There are numerous examples for *transport and flow* phenomena on biomembranes or bubble surfaces [1,2]. Examples in *structures* are membranes and shells [3,4].

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Phenomena on surfaces may also be coupled to processes in the surrounding volume such as in surfactant transport, hydraulic fracturing, reinforced structures etc. As an additional challenge, the surfaces may be moving [5–7], i.e. the domain of interest changes. The modeling of such phenomena naturally leads to boundary value problems where partial differential equations are formulated on manifolds. For the solution of such models, customized numerical methods are needed.

The first application of the finite element method for the solution of the Laplace–Beltrami operator on manifolds is reported in 1988 by Dziuk [8]. Since then, the topic has attracted a tremendous research interest leading to a variety of numerical methods for PDEs on surfaces existing today, see [9] for an overview. The most straightforward approach is to generate surface meshes on the manifold and extend the finite element method in a natural way using tangential differential calculus. That is, standard gradients of the planar two-dimensional case are replaced by surface gradients on the manifold. It is interesting to note that this approach has been chosen from the beginning in the simulation of transport phenomena, e.g. [10–12,9]. However, for the modeling of membranes and shells, a less intuitive path using local coordinate systems and Christoffel symbols is standard since a long time [3,4]. It is rather recent that these models have been recasted in the frame of global tangential operators [13–16].

Another approach is to only employ an implicit description of the manifold and solve the model equations on *all* iso-surfaces at once [17,18]. Then, the problem is naturally set up in the three-dimensional space embedding the manifolds, i.e. volumetric elements and shape functions are employed. However, typically only the solution on *one* iso-surface, say the zero-isosurface, is of interest. One may then restrict the surrounding domain to a narrow band around the manifold [10,19,20]. There are interesting similarities to phase field and diffuse interface approaches [21]. A recent approach is to collapse the narrow band to the manifold itself. Then, shape functions of the volumetric background elements are used, however, the integration takes place on the trace of the manifold only [22,23,15]. The resulting approaches are labeled TraceFEM [24,25,22,26,27] or CutFEM [28,15]. Higher-order approximations of PDEs on manifolds have been reported in different contexts before: For explicit handcrafted surface meshes in [29] and in the context of the TraceFEM in [27]. Adaptivity is considered e.g. in [30,31].

Herein, we propose a higher-order accurate approach for the approximation of PDEs on manifolds. The manifold is described implicitly based on the level-set method. A surface mesh composed by mixed higher-order quadrilateral and triangular finite elements is automatically generated from a background mesh and given level-set functions. A master level-set function defines the shape of the manifold. However, as the implied zero-isosurface may be infinite, it is restricted by additional (slave) level-set functions. That is, several level-set functions imply the bounded manifold being the domain of interest in the BVP. As a model problem, we consider the Laplace–Beltrami operator and an instationary advection–diffusion problem. Based on this, the extension of the approach to more advanced transport problems on surfaces and in the simulation of membranes and shells will be reported in forthcoming publications.

The automatic detection of higher-order surface elements has been reported by the authors in [32,33] in the context of integration and interpolation. There, only a set of surface elements is needed featuring double nodes and not necessarily fulfilling C_0-continuity. In order to be suited for the approximation of PDEs on surfaces as discussed herein, (1) continuity requirements have to be fulfilled, (2) the elements must be sufficiently shape regular, and (3) connectivity information in the usual FEM sense has to be provided, enabling the concept of nodal degrees of freedom. These issues are addressed herein with emphasis on higher-order accurate approximations. Also, the concept of using several level-set functions for the definition of the bounded manifold is new and an extension of [33].

The paper is organized as follows: In Section 2 we outline the geometric description of the bounded manifold based on several level-set functions defined on a background mesh composed by higher-order elements. The automatic generation of suitable higher-order surface meshes is described in Section 3: The reconstruction of surface elements approximating the zero-isosurface of the master level-set function, the restriction by means of additional (slave) level-set functions, the extraction of a continuous surface mesh from the element set, and the manipulation of the background mesh to achieve shape regular elements. Section 4 shortly recalls the standard finite element approach for approximations on meshed surfaces. Numerical results are presented in Section 5 for curved lines in two dimensions and curved surfaces in three dimensions. The Laplace–Beltrami operator is considered as well as instationary advection–diffusion on manifolds. Finally, a summary and outlook is given in Section 6.

2. Preliminaries

The task is to solve a boundary value problem (BVP) on a surface Γ in three dimensions. Let the surface be possibly curved, sufficiently smooth, orientable, connected (so there is only *one* surface), and feature a finite fixed area. The