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A stabilized cut finite element method for the Darcy problem on surfaces

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Highlights

- A new unfitted finite element method for the Darcy problem on surfaces is developed.
- Two novel stabilizations render method insensitve to surface position in the mesh.
- A priori error estimates including velocity, pressure and geometry error are given.
- Proposed normal gradient stabilization makes high order discretization possible.
- Tangential conditions for the velocity and pressure gradient are weakly enforced.

Abstract

We develop a cut finite element method for the Darcy problem on surfaces. The cut finite element method is based on embedding the surface in a three dimensional finite element mesh and using finite element spaces defined on the three dimensional mesh as trial and test functions. Since we consider a partial differential equation on a surface, the resulting discrete weak problem might be severely ill conditioned. We propose a full gradient and a normal gradient based stabilization computed on the background mesh to render the proposed formulation stable and well conditioned irrespective of the surface positioning within the mesh. Our formulation extends and simplifies the Masud–Hughes stabilized primal mixed formulation of the Darcy surface problem proposed in Hansbo and Larson (2016) on fitted triangulated surfaces. The tangential condition on the velocity and the pressure gradient is enforced only weakly, avoiding the need for any tangential projection. The presented numerical analysis accounts for different polynomial orders for the velocity, pressure, and geometry approximation which are corroborated by numerical experiments. In particular, we demonstrate both theoretically and through numerical results that the normal gradient stabilized variant results in a high order scheme.
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Keywords: Surface PDE; Darcy problem; Cut finite element method; Stabilization; Condition number; A priori error estimates

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1. Introduction

1.1. Background and earlier work

In recent years, there has been a rapid development of cut finite element methods, also called trace finite element methods, for the numerical solution of partial differential equations (PDEs) on complicated or evolving surfaces embedded into $\mathbb{R}^d$. The main idea is to use the restriction of finite element basis functions defined on a d-dimensional background mesh to a discrete, piecewise smooth surface representation which is allowed to cut through the mesh in an arbitrary fashion. The active background mesh then consists of all elements which are cut by the discrete surface, and the finite element space restricted to the active mesh is used to discretize the surface PDE. This approach was first proposed in [1] for the Laplace–Beltrami on a closed surface, see also [2] and the references therein for an overview of cut finite element techniques.

Depending on the positioning of the discrete surface within the background mesh, the resulting system matrix might be severely ill conditioned and either preconditioning [3] or stabilization [4] is necessary to obtain a well conditioned linear system. The stabilization introduced and analyzed in [4] is based on so called face stabilization or ghost penalty, which provides control over the jump in the normal gradient across interior faces in the active mesh. In particular, it was shown that the condition number scaled in an optimal way, independent of how the surface cut the background mesh. Thanks to its versatility, the face based stabilization can naturally be combined with discontinuous cut finite element methods as demonstrated in [5]. To reduce the matrix stencil and ease the implementation, a particular simple low order, full gradient stabilization using continuous piecewise linears was developed and analyzed in [6] for the Laplace–Beltrami operator. Then a unifying abstract framework for analysis of cut finite element methods on embedded manifolds of arbitrary codimension was developed in [7] and, in particular, the normal gradient stabilization term was introduced and analyzed. Further developments include convection problems [8,9], coupled bulk-surface problems [10,11] and higher order versions of trace fem for the Laplace–Beltrami operator [12,13]. Moreover, extensions to time-dependent, parabolic-type problems on evolving domains were proposed in [14,15].

So far, with their many applications to fluid dynamics, material science and biology, e.g., [16–21], mainly scalar-valued, second order elliptic and parabolic type equations have been considered in the context of cut finite element methods for surface PDEs. Only very recently, vector-valued surface PDEs in combination with unfitted finite element technologies have been considered, for instance in the numerical discretization of surface-bulk problems modeling flow dynamics in fractured porous media [22–25]. But while these contributions employed cut finite element type methods to discretize the bulk equations, only fitted (mixed and stabilized) finite elements methods on triangulated surfaces have been developed for vector surface equation such as the Darcy surface problem, see for instance [26,27]. The present contribution is the first where a cut finite element method for a system of partial differential equations on a surface involving tangent vector fields of partial differential equations is developed.

1.2. New contributions

We develop a stabilized cut finite element method for the numerical solution of the Darcy problem on a surface. The proposed variational formulation follows the approach in [27] for the Darcy problem on triangulated surfaces which is based on the stabilized primal mixed formulation by Masud and Hughes [28]. Note that standard mixed type elements are typically not available on discrete cut surfaces. Combining the ideas from [27] with the stabilized full gradient formulations of the Laplace–Beltrami problem from [6,7], the tangent condition on both the velocity and the pressure gradient is enforced only weakly. When employing finite element function from the full d-dimensional background mesh, such a weak enforcement of the tangential condition is convenient and rather natural.

To render the proposed formulation stable and well conditioned irrespective of the relative surface position in the background mesh, we consider two stabilization forms: the full gradient stabilization introduced in [6] which is convenient for low order elements, and the normal gradient stabilization introduced in [7] which also works for higher order elements. Through these stabilizations, we gain control of the variation of the solution orthogonal to the surface, which in combination with the Masud–Hughes variational formulation results in a coercive formulation of the Darcy surface problem. In practice, the exact surface is approximated leading to a geometric error which we take into account in the error analysis. We show stability of the method and establish optimal order a priori error estimates. The presented numerical analysis also accounts for different polynomial orders for the velocity, pressure, and geometry approximation.

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