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Stability of peridynamic correspondence material models and their particle discretizations

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Abstract

Peridynamic correspondence material models provide a way to combine a material model from the local theory with the inherent capabilities of peridynamics to model long-range forces and fracture. However, correspondence models in a typical particle discretization suffer from zero-energy mode instability. These instabilities are shown here to be an aspect of material stability. A stability condition is derived for state-based materials starting from the requirement of potential energy minimization. It is shown that all correspondence materials fail this stability condition due to zero-energy deformation modes of the family. To eliminate these modes, a term is added to the correspondence strain energy density that resists deviations from a uniform deformation. The resulting material model satisfies the stability condition while effectively leaving the stress tensor unchanged. Computational examples demonstrate the effectiveness of the modified material model in avoiding zero-energy mode instability in a peridynamic particle code.

Keywords: Particle methods; Meshless methods; Peridynamic; Nonlocal; Elasticity

1. Introduction

Peridynamics is a nonlocal theory of mechanics in which material points in a continuum or in a group of discrete particles interact with each other through forces. The exchange of forces between any pair of material points is called a *bond interaction*. These forces are determined by the material model according to the current deformation.

Material models in peridynamics explicitly determine the force in each bond. Many material models have been investigated as the peridynamic theory has been developed. A particularly useful class of material models is called the *correspondence* models. These models use a stress–strain relation from the local theory as a means to determine the bond forces. In principle, any material model from a finite element code can be applied directly in a peridynamic computational simulation when formulated as a correspondence model.

Correspondence material models have been used successfully in the modeling of a wide variety of material response, including elastic–plastic and viscoplastic materials with large strains and fracture, for example [1–3].

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However, the usefulness of peridynamic correspondence models has been impeded by some practical issues in their implementation in meshless peridynamic codes. One such issue is the difficulty that is sometimes encountered when evaluating the *shape tensor* defined in Eq. (15), particularly when very large deformations or extensive damage are present. A more troublesome issue with correspondence models is their tendency to exhibit zero-energy mode oscillations in meshless simulations. These oscillations typically build up gradually during a long simulation, but they can eventually ruin the results. They can occur even when the appropriate version [4] of the Courant–Friedrichs–Levy time step restriction is applied. The purpose of the present paper is to investigate the root cause of this type of instability as a manifestation of a *material* instability, rather than merely an artifact of the meshless discretization. A stabilized correspondence material model is proposed that is stable in both the continuum and discretized equations.

2. Peridynamics background

Peridynamics is a generalization of the standard theory of solid mechanics. It treats the mechanics of continuous bodies and discrete particles, including long-range forces and fracture, within the same basic field equations. A typical material point $\mathbf{x}$ in a body $\mathcal{B}$ interacts with other material points $\mathbf{q}$ within a neighborhood called the *family* of $\mathbf{x}$, denoted $\mathcal{H}$. The radius of the neighborhood is called the *horizon*, δ. The vector in the undeformed configuration from $\mathbf{x}$ to any of its neighbors $\mathbf{q} \in \mathcal{H}$ is called a *bond*, generically given the symbol $\boldsymbol{\xi} = \mathbf{q} - \mathbf{x}$. The vector $\boldsymbol{\xi} = \mathbf{0}$ is excluded from the family.

The equation of motion in peridynamics is given by

$$\rho(\mathbf{x})\ddot{\mathbf{y}}(\mathbf{x}, t) = \int_{\mathcal{H}} \mathbf{f}(\mathbf{q}, \mathbf{x}, t)\, dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x}, t) \quad \forall \mathbf{x} \in \mathcal{B},\ t \geq 0, \tag{1}$$

where ρ is the density, $\mathbf{y}$ is the deformation, and $\mathbf{b}$ is the prescribed body force density field. The equation of equilibrium (see Eq. (24) for the derivation) is written

$$\int_{\mathcal{H}} \mathbf{f}(\mathbf{q}, \mathbf{x})\, dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x}) = \mathbf{0} \quad \forall \mathbf{x} \in \mathcal{B}. \tag{2}$$

The vector-valued function $\mathbf{f}(\mathbf{q}, \mathbf{x}, t)$ is the *pairwise bond force density* (with units of force per volume squared) that $\mathbf{q}$ exerts on $\mathbf{x}$. The material model determines the values of $\mathbf{f}$ according to the deformations of the families of $\mathbf{x}$ and of $\mathbf{q}$.

The simplest class of material model [5] is called *bond-based*, in which $\mathbf{f}(\mathbf{q}, \mathbf{x}, t)$ depends *only* on the deformation of the bond $\mathbf{q} - \mathbf{x}$. More general material response can be incorporated using the *state-based* class of models [6], in which $\mathbf{f}(\mathbf{q}, \mathbf{x}, t)$ depends not only on $\mathbf{q} - \mathbf{x}$, but also on the deformation of all the other bonds in $\mathcal{H}$. State-based material models are expressed using mathematical objects called *states*, which are mappings from $\mathcal{H}$ to some other quantity, either vector- or scalar-valued. By convention, the bond that a state $\underline{\mathbf{A}}$ operates on is written in angle brackets, $\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle$. Dependence of the state on position and time is written in square brackets, $\underline{\mathbf{A}}[\mathbf{x}, t]$.

The *deformation state* $\underline{\mathbf{Y}}$ maps bonds onto their deformed images:

$$\underline{\mathbf{Y}}[\mathbf{x}, t]\langle\boldsymbol{\xi}\rangle = \mathbf{y}(\mathbf{x} + \boldsymbol{\xi}, t) - \mathbf{y}(\mathbf{x}, t). \tag{3}$$

The bond force density $\mathbf{t}(\mathbf{q}, \mathbf{x}, t)$ that $\mathbf{q}$ exerts on $\mathbf{x}$ due to the material model at $\mathbf{x}$ is expressed in the *force state* $\underline{\mathbf{T}}$,

$$\underline{\mathbf{T}}[\mathbf{x}, t]\langle\mathbf{q} - \mathbf{x}\rangle = \mathbf{t}(\mathbf{q}, \mathbf{x}, t). \tag{4}$$

Due to the requirement for balance of linear momentum, the pairwise bond force density contains contributions from the material models at both $\mathbf{x}$ and $\mathbf{q}$:

$$\begin{aligned} \mathbf{f}(\mathbf{q}, \mathbf{x}, t) &= \mathbf{t}(\mathbf{q}, \mathbf{x}, t) - \mathbf{t}(\mathbf{x}, \mathbf{q}, t) \\ &= \underline{\mathbf{T}}[\mathbf{x}, t]\langle\mathbf{q} - \mathbf{x}\rangle - \underline{\mathbf{T}}[\mathbf{q}, t]\langle\mathbf{x} - \mathbf{q}\rangle. \end{aligned} \tag{5}$$

A *material model* $\hat{\underline{\mathbf{T}}}$ is a function that maps a deformation state onto a force state:

$$\underline{\mathbf{T}}[\mathbf{x}, t] = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}[\mathbf{x}, t]).$$

For example, a bond-based material model might have the form

$$\hat{\underline{\mathbf{T}}}\langle\boldsymbol{\xi}\rangle = \frac{1}{2} C(\boldsymbol{\xi}) \underline{\mathbf{M}}\langle\boldsymbol{\xi}\rangle \frac{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle| - |\boldsymbol{\xi}|}{|\boldsymbol{\xi}|}, \qquad \underline{\mathbf{M}}\langle\boldsymbol{\xi}\rangle = \frac{\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|}, \tag{6}$$