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A conservative, second order, unconditionally stable artificial compression method

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Highlights

- Introduces an unconditionally stable, second order artificial compression method.
- Proves unconditional, long time, nonlinear stability of the method.
- Shows the method is exactly conservative in the appropriate context.
- Analyzes nonphysical acoustic waves generated by artificial compressibility.
- Analyzes the nonphysical, nonlinear acoustic sound source and shows it is small.
- Provides accuracy tests that indicate convergence as predicted by the theory.
- Provides summary conclusions and open problems.

Abstract

This report presents a new artificial compression method for incompressible, viscous flows. The method has second order consistency error and is unconditionally, long time, energy stable for the velocity and, weighted by the timestep, for the pressure. It uncouples the pressure and velocity and requires no artificial pressure boundary conditions. When the viscosity $\nu = 0$ the method also exactly conserves a system energy. The method is based on a Crank–Nicolson Leapfrog time discretization of the slightly compressible model

$$
\begin{aligned}
&(1 - \varepsilon_1 \operatorname{grad}\operatorname{div})u_t + u \cdot \nabla u + \frac{1}{2}(\operatorname{div} u)u - \nu \Delta u + \nabla p = f \\
&\quad \text{and} \;\; \varepsilon_2 p_t + \operatorname{div} u = 0.
\end{aligned}
$$

This report presents the method, gives a stability analysis, presents numerical tests and gives a preliminary analysis with tests of the non-physical acoustic waves generated. Consideration of the physical fidelity of the artificial compression method leads to a related method.

Keywords: Artificial compression; Crank–Nicolson; Leapfrog

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1. Introduction

Consider the time dependent incompressible Navier–Stokes equations in a $2d$ or $3d$ domain Ω for the fluid velocity and pressure, $u(x,t)$, $p(x,t)$:

$$u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = f \text{ and } \operatorname{div} u = 0 \text{ in } \Omega \times (0,T], \tag{1.1}$$
$$\int_\Omega p(x,t)\,dx = 0,\, u = 0, \text{ on } \partial\Omega \text{ and } u(x,0) = u_0(x).$$

Respectively, ν, f, u_0 are the kinematic viscosity, body force and initial velocity.

As problems become larger and assessment of uncertainty becomes necessary, execution time can become of primary importance. Execution time limitations often force the coupling between the velocity and pressure to be broken in various ways including adding a small (artificial) compression term, studied herein. Doing so speeds up the computations dramatically and does not require pressure boundary conditions but introduces extra numerical errors and new physical flow behaviors associated with compressibility, e.g., [1]. These include non-physical fast pressure oscillations (acoustics), analyzed in Section 3. These fast acoustic waves can yield restrictive timestep conditions for explicit time discretization of the pressure equation. The method presented below is a second order, artificial compression method with explicit treatment of the pressure that is, nevertheless, *unconditionally* stable.

Since the velocity–pressure uncoupling is through the time discretization and applicable to other space discretizations, we suppress the (secondary) spacial discretization. In the tests in Sections 3 and 4, a standard finite element method is used for spacial discretization.

Algorithm 1.1 (*Artificial Compression Method*)**.** Given time step $k > 0$, $t_n = nk$, and $u_n(x) \cong u(x,t_n)$, $p_n(x) \cong p(x,t_n)$. Pick $\alpha, \beta > 0$ constants with

$$\alpha\beta \geq \frac{1}{4},$$

Let either $u_n^* = u_n$ or

$$u_n^* = 2\frac{u_n + u_{n-2}}{2} - \frac{u_{n-1} + u_{n-3}}{2}. \tag{1.2}$$

Given (u_n, p_n), (u_{n-1}, p_{n-1}), find (u_{n+1}, p_{n+1}) satisfying:

$$\begin{aligned}
&\frac{u_{n+1} - u_{n-1}}{2k} - \beta k^{-1} \operatorname{grad} \operatorname{div}(u_{n+1} - u_{n-1}) + \\
&+ u_n^* \cdot \nabla \left(\frac{u_{n+1} + u_{n-1}}{2}\right) + \frac{1}{2}(\operatorname{div} u_n^*)\left(\frac{u_{n+1} + u_{n-1}}{2}\right) + \\
&- \nu\Delta(\frac{u_{n+1} + u_{n-1}}{2}) + \nabla p_n = f(x,t_n), \\
&\alpha k(p_{n+1} - p_{n-1}) + \operatorname{div} u_n = 0, \\
&u_{n+1} = 0 \text{ on } \partial\Omega \text{ and } \int_\Omega p_{n+1} dx = 0.
\end{aligned} \tag{1.3}$$

The roles played by the parameters α (units $1/L^2$), β (units L^2) are as follows. $2\alpha k^2$ is the standard artificial compression parameter that allows the pressure to be advanced explicitly in time. The 2β term is a dispersive regularization that acts through the momentum equation to ensure unconditional stability of the continuity equation, Remark 3.1. Thus if $\beta = 0$ the method would require a time step condition for stability. The term does increase the condition number and the coupling among velocity components in the linear system to be solved at each step. In $2d$ the increased coupling can be corrected using the sparse-grad–div adjustment of [2].

The nonlinearity is explicitly skew symmetrized in the second line of the algorithm and in the numerical tests in Section 4. For comparisons of different presentations of the nonlinearity, we refer to [3]. The extrapolation (1.2) is an idea of Ingram [4] with roots back to Baker [5]. Often uncoupling of velocity and pressure has been achieved by fractional step or operator splitting methods, e.g., [6–8]. Further efficiency gains per step can be obtained at a cost of an $\mathcal{R}e$ related time step condition by explicit discretization of the nonlinear terms; for interesting recent work in this approach see [9]. This method (1.3) is inspired by the original method of Chorin [10] and a recent pressure correction method connected with the stabilized implicit–explicit (IMEX) method CNLFstab of [11]. Artificial (or quasi- or