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A posteriori error estimation for the steady Navier–Stokes equations in random domains

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Highlights

- An analysis of the steady Navier–Stokes equations in random domains is performed.
- A method based on mapping to a reference domain combined with perturbation is proposed.
- A posteriori error estimates that control both finite element and perturbation errors are derived.

Abstract

We consider finite element error approximations of the steady incompressible Navier–Stokes equations defined on a randomly perturbed domain, the perturbation being small. Introducing a random mapping, these equations are transformed into PDEs on a fixed reference domain with random coefficients. Under suitable assumptions on the random mapping and the input data, in particular the so-called *small data* assumption, we prove the well-posedness of the problem. We assume then that the mapping depends affinely on L independent random variables and adopt a perturbation approach expanding the solution with respect to a small parameter ε that controls the amount of randomness in the problem. We perform an *a posteriori* error analysis for the first order approximation error, namely the error between the exact (random) solution and the finite element approximation of the first term in the expansion with respect to ε. Numerical results are given to illustrate the theoretical results and the effectiveness of the error estimators.

Keywords: a posteriori error estimate; Finite element method; Uncertainty quantification; Random domains; Navier–Stokes equations

1. Introduction

It is nowadays common to include uncertainty in the modelling of complex phenomena arising for instance in physics, biology or engineering to reflect an intrinsic variability of the system or our inability to adequately characterize the input data, due for instance to experimental measurements, such as the coefficients, forcing term, boundary conditions or geometry. The goal is then to determine the effect of the uncertainty on the solution or a specific quantity

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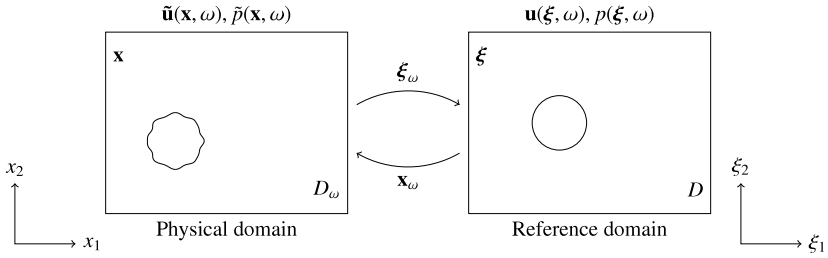

Fig. 1. Illustration and notation for the domain mapping approach.

of interest. In a probability setting, the uncertainty is characterized via random variables or more generally random fields, and yields to stochastic partial differential equations (SPDEs).

In this paper, we focus our study on the steady-state Navier–Stokes equations defined on a domain with randomly perturbed boundary, considering small perturbations. We assume that the random perturbation is given and reflects the uncertainty in the geometry, due for instance to manufacturing tolerances or material fatigue and obtained e.g. from experimental measurements or prior knowledge. For simplicity, we assume that the uncertainty in the system is only due to the random domain, but the analysis can be straightforwardly extended to include other sources of randomness. Moreover, we stress that all the analysis, namely the well-posedness of the problem (in particular the uniqueness) and the error analysis, is performed under the assumption of small data.

Several approaches have been developed to perform analysis and numerical approximation of PDEs in random domains, such as the fictitious domain method [1], the perturbation method based on shape calculus [2] and the domain mapping method initially proposed by [3] and also used for instance in [4–6]. In the first approach, the PDEs are extended to a fixed reference domain, the so-called fictitious domain, which contains all the random domains. The original boundary condition is then imposed through a Lagrange multiplier yielding a saddle-point problem to be solved in the fictitious domain. In the perturbation method, which is suitable for small perturbations only, the solution is represented using a shape Taylor expansion with respect to the (random) perturbation field of the boundary of the domain. Finally, the domain mapping approach, which is the one considered in this work, transforms the deterministic PDEs defined on a random domain into PDEs on a fixed reference domain with random coefficients via a random mapping. We give on Fig. 1 an illustration of the mapping for a given ω between the physical domain and the reference one, supplemented with some notation. Contrary to the method based on shape derivatives, our approach requires the construction of a mapping defined in the whole domain consistent with the random perturbation of the boundary. If the random mapping is not given analytically, it can be obtained by solving appropriate equations, e.g. Laplace equation as it is done in [3]. The domain mapping method prevents the need of remeshing and can make use of the well-developed theory for PDEs with random coefficients (random PDEs in short) on deterministic domains. Numerical approximations of random PDEs on the fixed reference domain can indeed be obtained through any of the well-known techniques, such as Monte-Carlo methods [7] and their generalizations as quasi-Monte Carlo [8–10] and multi-level Monte-Carlo [11–13], or the stochastic spectral methods comprising the Stochastic Galerkin [14,15] and the Stochastic Collocation [16–18] methods.

In this work, once the PDEs are transformed on the reference domain, we proceed as in [19] and use a perturbation approach [20] expanding the exact random solution with respect to a parameter ε that controls the level of uncertainty in the problem. This approach yields uncoupled deterministic problems for each term in this expansion, which can be solved using for instance the finite element (FE) method. The main goal of this paper is to perform an *a posteriori* error analysis for the error between the exact random solution and the finite element approximation of the first term in the expansion, that is the solution corresponding to the case $\varepsilon = 0$. The two error estimators we obtain are constituted of two parts, namely one part η_h due to the physical space discretization and another one η_ε due to the uncertainty. Such *a posteriori* error estimators can be used for instance to determine a spatial mesh that yields a numerical accuracy comparable with the model uncertainty, i.e. such that η_h and η_ε are balanced, proceeding either by uniform or adaptive refinements of the mesh. We stress that the computation of the first error estimator requires only the FE approximation of the solution of the problem for $\varepsilon = 0$ and the Jacobian matrix of the mapping between the reference domain and the physical random domain.