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Dimensional hyper-reduction of nonlinear finite element models via empirical cubature

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Abstract

We present a general framework for the dimensional reduction, in terms of number of degrees of freedom as well as number of integration points ("hyper-reduction"), of nonlinear parameterized finite element (FE) models. The reduction process is divided into two sequential stages. The first stage consists in a common Galerkin projection onto a reduced-order space, as well as in the condensation of boundary conditions and external forces. For the second stage (reduction in number of integration points), we present a novel cubature scheme that efficiently determines optimal points and associated positive weights so that the error in integrating reduced internal forces is minimized. The distinguishing features of the proposed method are: (1) The minimization problem is posed in terms of orthogonal basis vector (obtained via a *partitioned* Singular Value Decomposition) rather that in terms of snapshots of the integrand. (2) The volume of the domain is exactly integrated. (3) The selection algorithm need not solve in all iterations a nonnegative least-squares problem to force the positiveness of the weights. Furthermore, we show that the proposed method converges to the absolute minimum (zero integration error) when the number of selected points is equal to the number of internal force modes included in the objective function. We illustrate this model reduction methodology by two nonlinear, structural examples (quasi-static bending and resonant vibration of elastoplastic composite plates). In both examples, the number of integration points is reduced three order of magnitudes (with respect to FE analyses) without significantly sacrificing accuracy.

Keywords: Reduced-order model; Hyper-reduction; Optimized cubature; Finite elements; Singular Value Decomposition

1. Introduction

Generally speaking, model order reduction refers to any endeavor aimed at constructing a simpler model from a more complex one. The simpler model is usually referred to as the *reduced-order model* (ROM), while the more

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complex one is termed the full-order or high-fidelity model. This full-order model may be, for instance, – as is the case here – a finite element (FE) model.

The focus of the present paper is on the so-called *projection-based, reduced-order models*. The existence of such low-dimensional representations for a given parametrized finite element problem relies on the premise that the state variable can be accurately approximated by a linear combination of a few global basis vectors. The most common approach is to determine these basis vectors by applying some type of dimensionality reduction strategy (such as the Proper Orthogonal Decomposition, POD) over a so-called *training sample*. This sample is obtained by previously solving – in an *offline* stage – the full-order model for judiciously chosen values of the input parameters.

1.1. Approximation of nonlinear terms

In the general case of governing equations featuring terms that bear a nonaffine relationship with both the state variable and input parameters, the construction of an inexpensive low-dimensional model entails two sequential stages [1], namely: (1) projection onto the reduced basis, and (2) approximation of the nonlinear term. Once a basis matrix for the state variable is available, the projection stage is a standard operation consisting in introducing the approximation of the state variables in the governing equation, and then in posing the resulting equation in the space spanned by the basis vectors. This operation naturally leads to a significant reduction in the number of unknowns, and hence diminishes considerably the equation solving effort. However, in a general nonlinear case, the computational cost of evaluating the residual still depends on the size of the underlying finite element mesh—hence the need for a second reduction stage.

In contrast to the first reduction stage, which is more or less standard, the second stage of dimensionality reduction – Ryckelynck [2] coined the term *hyper-reduction* to refer to it – is far more challenging and still remains an issue of discussion in the model reduction community. In the following, we examine the various approaches encountered in the related literature to deal with this additional dimensionality reduction stage.

1.2. Classification of "hyper-reduction" methods

Let $\boldsymbol{F}^h \in \mathbb{R}^N$ denote[1] the full-order term bearing a general, nonaffine relationship with both the input variable and the state variable (in the context of this paper, $\boldsymbol{F}^h \in \mathbb{R}^N$ will be the vector of FE *nodal* internal forces). The corresponding projection onto the reduced order space will be represented by $\boldsymbol{F} \in \mathbb{R}^n$ ($n \ll N$), the connection between these two variables being the matrix of basis vectors $\boldsymbol{\Phi} \in \mathbb{R}^{N \times n}$ ($\boldsymbol{F} = \boldsymbol{\Phi}^T \boldsymbol{F}^h$). Existing approaches for dealing with the approximation of $\boldsymbol{F}$ can be broadly classified as *nodal vector approaches and integral approaches*.

1.2.1. Nodal vector approximation approaches ("gappy" data)

In this type of approaches, the approximation is carried out by replacing the finite element vector $\boldsymbol{F}^h$ by a low-dimensional interpolant $\boldsymbol{F}^h \approx \boldsymbol{R_F} \boldsymbol{F}^h_{\mathbf{z}}$, $\boldsymbol{R_F} \in \mathbb{R}^{N \times m}$ being the interpolation matrix, and $\boldsymbol{F}^h_{\mathbf{z}}$ the entries of $\boldsymbol{F}^h$ corresponding to the degrees of freedom ($\mathbf{z} \subset \{1, 2, \dots, N\}$) at which the interpolation takes place. The interpolation matrix is obtained following the common procedure of computing a basis matrix for $\boldsymbol{F}^h$, and then determining a set of indices so that the error is minimized over a set of representative snapshots of $\boldsymbol{F}^h$. This set of interpolation indices can be determined *offline* using procedures such as the Empirical Interpolation Method (EIM) [3,4], the Best Points Interpolation Method (BPIM) [5], the Discrete BPIM [6] or the Missing Point Estimation Method [7]. The idea behind this vector approximation approach has its roots in the landmark work of Everson and Sirovich [8] for reconstruction of "gappy" data, and was historically the first proposal for dealing with nonlinear terms in model order reduction; it has been adopted by, among others, [1,9–14]. Alternatively, [2] proposes to bypass the construction of the low-dimensional interpolant and simply solve the balance equations at appropriately selected degrees of freedom (collocation).

[1] A word in notation is in order here. The superindex h is employed throughout the paper to denote finite element nodal quantities; bare symbols, on the other hand, are associated to reduced-order variables, that is, variables projected onto the reduced-order space. Likewise, FE and reduced-order dimensions are represented by upper-case and lower-case symbols, respectively. For instance, N and n denote the number of unknowns in the FE and reduced-order problems, respectively, whereas M and m represents the total number of integration points in the FE and reduced-order problem, respectively.