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Worst-case multi-objective error estimation and adaptivity

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Abstract

This paper introduces a new computational methodology for determining a-posteriori multi-objective error estimates for finite-element approximations, and for constructing corresponding (quasi-)optimal adaptive refinements of finite-element spaces. As opposed to the classical goal-oriented approaches, which consider only a single objective functional, the presented methodology applies to general closed convex subsets of the dual space and constructs a worst-case error estimate of the finite-element approximation error. This worst-case multi-objective error estimate conforms to a dual-weighted residual, in which the dual solution is associated with an approximate supporting functional of the objective set at the approximation error. We regard both standard approximation errors and data-incompatibility errors associated with incompatibility of boundary data with the trace of the finite-element space. Numerical experiments are presented to demonstrate the efficacy of applying the proposed worst-case multi-objective error estimate in adaptive refinement procedures.

Keywords: A-posterior error estimation; Worst-case multi-objective error estimation; Adaptive finite-element methods

1. Introduction

Goal-oriented *a-posteriori* error-estimation and adaptivity has emerged over the past years as an effective methodology to solve measurement problems in computational science and engineering. As opposed to the classical norm-oriented adaptive methods that originated from the seminal work of Babuška and Rheinboldt [1,2], which aim to construct finite-element approximations that are optimal in an appropriate norm, goal-oriented adaptive methods seek to construct finite-element approximations that yield optimal approximations of a particular functional of the solution. Goal-oriented methods date back to the pioneering work of Becker and Rannacher [3,4], Oden and Prudhomme [5,6] and Giles, Süli, Houston and Hartmann [7–10]. It is noteworthy, however, that the use of dual solutions in a-posteriori error estimates, which forms the basis of goal-oriented error estimation, had already been pursued before by Johnson, Eriksson and Hansbo [11–14]. Goal-oriented adaptivity has been successfully applied to a wide variety of problems including compressible and incompressible flow problems [10,15,16], elasticity and plasticity [17–19], variational

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inequalities [20], fluid–structure interaction [21–24], free-boundary problems [25,26], phase-field models [27,28], and kinetic equations [29]. This overview is in fact far from exhaustive, and many more applications and contributions can be found in the literature. It is notable, however, that despite the success of goal-oriented methods in many applications, convergence and optimality theory is still rudimentary; see [30,31].

In addition to goal-oriented adaptivity of the mesh width (h) or local order (p) of finite-element approximation spaces, goal-oriented approaches have also been considered in the context of model adaptivity. In these approaches, the error indicator is applied to systematically decide between a simple coarse model and a complex sophisticated model, in such a manner that the sophisticated model is only applied in regions of the domain that contribute most significantly to the objective functional under consideration. For examples of goal-oriented model adaptivity, we refer to [32] for application of goal-oriented model adaptivity to heterogeneous materials, to [33] for goal-oriented atomistic/continuum adaptivity in solid materials, and to [34] for goal-oriented adaptivity between a boundary-integral formulation of the Stokes equations and a PDE formulation of the Navier–Stokes equations.

The premise in goal-oriented strategies is that interest is restricted to a single functional output, rather than all details of the solution of the system under consideration. Although there are indeed many applications in which interest is restricted to a single scalar output of a system, there are in fact many more in which the object of interest is not so narrowly defined. There are for instance many applications in which multiple scalar outputs are of interest, e.g. the aerodynamic drag, lift and torques that are exerted by a flow on an immersed object [35]. Another example is furnished by the many cases in which one is concerned with the solution in a particular region of the problem domain. This scenario also covers boundary-coupled multi-physics problems, e.g. fluid–structure interaction [21], in which two subsystems interact via a mutual interface and only one of the subsystems is of importance, and free-boundary problems [25,26], in which interest extends to the geometry of the domain only and not to the solution of the underlying partial differential equation, or vice versa. A similar situation arises for volumetrically-coupled multi-physics problems, e.g. electro-thermo-mechanical problems [36], if significance is assigned to one field only. Another pertinent class of problems that are incompatible with the premise of goal-oriented strategies, pertains to applications in which one is interested in the flux functional on a part of the boundary that is subjected to essential boundary conditions or, reciprocally, the trace of the solution on a part of the boundary that is furnished with natural boundary conditions. In all of the aforementioned cases, interest does not extend to all details of the solution, but it is also not restricted to a single quantity. Instead, the objective set in these cases corresponds to a proper non-singleton subset of the dual space, and not to an individual element of the dual space as in goal-oriented approaches.

The purpose of this paper is to introduce a new computational methodology for determining a-posteriori *multi-objective* error estimates for finite-element approximations, and for constructing corresponding (quasi-)optimal adaptive refinements of finite-element spaces. Such multi-objective error estimation and adaptivity has only received scant consideration so far; see [37,35]. As opposed to the aforementioned approaches, our methodology applies to generic possibly infinite-dimensional closed convex subsets of the dual space. The presented methodology relies on the construction of a worst-case error estimate, viz. the supremum of the error over the objective set. This worst-case error corresponds to the value of the supporting function of the objective set at the finite-element approximation error. We then construct an approximate supporting functional of the objective set at the approximation error, and determine an approximate dual solution for this supporting functional to form an error estimate in dual-weighted-residual form [3]. The corresponding error estimate represents a *worst-case multi-objective error estimate* for the finite-element approximation on the objective set. In this work, we regard both standard approximation errors and data-incompatibility errors associated with incompatibility of boundary data with the trace of the finite-element space.

The presented worst-case multi-objective error estimate is based on a technique for estimating linear functionals of solutions of equations with unbounded closed linear operators set in Hilbert spaces [38]. This technique applies to general linear operators with non-trivial null-spaces and possibly non-closed ranges, and it is based on an extension of Young–Fenchel duality to closed unbounded linear operators [39,38]. It is noteworthy that this technique for estimating a linear functional of a solution of an operator equation, which we consider in this paper in the context of worst-case multi-objective error estimation, yields a natural generalization of adaptive-state-estimation or so-called data-assimilation procedures (such as online sequential filters [40–42] or offline variational approaches [43]) to a wide class of linear and nonlinear partial differential equations, enabling incorporation of a-posteriori knowledge (for instance, sensor readings) into the error estimate rendering it less conservative [41,44].