Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 318 (2017) 937–956
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Divide and conquer: An incremental sparsity promoting compressive sampling approach for polynomial chaos expansions

Negin Alemazkoor, Hadi Meidani*

Department of Civil and Environmental Engineering, University of Illinois at Urbana-Champaign, Urbana, IL, USA

Received 22 June 2016; received in revised form 2 December 2016; accepted 31 January 2017
Available online 20 February 2017

Highlights

- The proposed sparse recovery promotes the sparsity by breaking the dimensionality.
- Sparsest polynomial expansion is sought through combinations of dimension and order.
- The algorithm decreases prediction error and reduces the dimensionality.
- It is particularly successful when a lower dimensional accurate model exits.
- It outperforms conventional compressive sampling approaches for polynomial expansions.

Abstract

This paper introduces an efficient sparse recovery approach for Polynomial Chaos (PC) expansions, which promotes the sparsity by breaking the dimensionality of the problem. The proposed algorithm incrementally explores sub-dimensional expansions for a sparser recovery, and shows success when removal of uninfluential parameters that results in a lower coherence for measurement matrix, allows for a higher order and/or sparser expansion to be recovered. The incremental algorithm effectively searches for the sparsest PC approximation, and not only can it decrease the prediction error, but it can also reduce the dimensionality of PCE model. Four numerical examples are provided to demonstrate the validity of the proposed approach. The results from these examples show that the incremental algorithm substantially outperforms conventional compressive sampling approaches for PCE, in terms of both solution sparsity and prediction error.

Keywords: Compressive sampling; Polynomial chaos expansion; Uncertainty quantification; Sparse approximation; Legendre polynomials

1. Introduction

To reliably predict the behavior of a physical system of interest, it is vital to accurately model the interaction between the system's inputs and outcomes, and account for the input uncertainty and its impact on the outcomes or the quantities of interests. The uncertain inputs to the model can include uncertain initial or boundary conditions,

* Corresponding author.
E-mail address: meidani@illinois.edu (H. Meidani).

http://dx.doi.org/10.1016/j.cma.2017.01.039
0045-7825/

uncertain parameters, etc., which are often modeled as random variables. Uncertainty quantification (UQ) approaches are used to model the propagation of random input variables through the model and quantify the uncertainty in the outputs. The most commonly known and used UQ approach is the Monte Carlo approach, which despite its simple and robust applicability, suffers from a poor convergence rate of $\mathcal{O}(M^{-\frac{1}{2}})$, where M is the number of samples. Although extensive research has been done and several modifications to the original Monte Carlo approach have been proposed to improve this convergence rate, e.g. importance sampling [1], quasi-Monte Carlo [2], multilevel Monte Carlo [3]) these sample-based approaches still suffer from slow rate of convergence. Therefore, there has been a growing interest in exploring alternative numerical approaches for UQ.

Stochastic spectral approaches, as one alternative, have been shown to offer accelerated convergence and speed-up over Monte Carlo approaches, thereby being recognized as a promising solution to a variety of engineering problems especially when the dimensionality of parameter space is not large. In these methods, the random output is represented through a spectral expansion on specific orthogonal basis functions such as generalized polynomial chaos (PC) [4,5]. There are two distinguished classes of methods to evaluate the coefficients of the polynomial expansion, stochastic Galerkin and stochastic collocation methods. Stochastic Galerkin method is an extension of Galerkin approach, in which continuous operator problems such as differential equations are converted to a system of equations. Similar to Galerkin scheme, stochastic Galerkin leads to a K-coupled equation system, where K is the number of coefficients to be estimated in polynomial chaos expansion (PCE), making it an intrusive approach. When the original problem has a complex nonlinear form, deriving the coupled equations can be cumbersome or even impossible.

Alternatively, in non-intrusive stochastic collocation methods, the coefficients are estimated either by solving a least square problem to fit a set of sample responses or by recasting the coefficients as integrals and approximating the integrals using sampling, quadrature, or sparse grid approaches. Despite the Galerkin method, the complexity of the original problem does not impact the applicability of collocation methods. However, due to the curse of dimensionality, when the number of uncertain input parameters is large, even when efficient techniques such as sparse grid [6–8] are used, prohibitively large number of sample points are required for accurate estimation of chaos coefficients. Adaptive methods such as adaptive sparse grid [9,10] and functional ANOVA decomposition [11,12] have been developed in order to reduce the required number of samples by identifying the unimportant dimensions and placing fewer sample points in those dimensions.

Recently, additional efforts have been made to explore the regularity of the solution using a small number of samples when the solution is sparse and its approximated PCE has only few terms. Specifically, compressive sampling was used in [13–15] as a non-intrusive non-adapted approach to approximate the sparse solution of stochastic problems. Compressive sampling was first initiated in the field of signal and image reconstruction [16–18]. Conventionally, based on the distinguished Shannon's theorem, sufficient sampling rate was considered to be larger than twice the maximum frequency in the signal [19]. Moreover, the fundamentals of linear algebra also impose the number of samples to be equal or larger than the dimensionality of the signal to ensure the reconstruction. However, compressive sampling techniques allowed reconstruction of sparse signals and images from incomplete measurements, i.e. a small set of samples, and effectively solving an underdetermined linear system [16–18].

In non-intrusive estimation of PCE, samples are acquired from an actual experimentation on the behavior of the system or from numerical simulations of a high-fidelity model for the system. Both experimentation and simulation of complex systems can be very expensive, calling for efforts to reduce the number of required samples in non-intrusive estimation procedures. In [13,14], compressive sampling was successfully applied in estimating the chaos coefficients where the number of samples was significantly smaller than the number of unknown chaos coefficients. However, this success heavily depends on two conditions: (a) the solution of stochastic problem should be in fact sufficiently sparse, and (b) the measurement matrix which includes random evaluations of polynomial bases should be sufficiently incoherent [13]. In [20], the authors focused on improving the first condition, i.e. the actual sparsity of the solution, by rotating the random inputs using the active subspace method [21]. As a result of this rotation, only a few influential bases will effectively participate in the spectral representation of the quantity of interest (QoI), making the target representation sparser. However, as the authors admit, the coordinate rotation has negligible impact on the coherence of the measurement matrix. This means that the second condition for a successful sparse recovery is not improved. That is, no matter how effectively one can transform the coordinate system of the random inputs to enable a sparser *target* solution, the attempts to recover that target solution using the available samples will still be in vain if the measurement matrix is highly coherent. Our proposed approach tries to be aware of this second condition on the coherence. Specifically, instead of rotating the uncertainty sources, it concerns selecting for the polynomial expansion