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A rigorous and unified mass lumping scheme for higher-order elements

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Highlights

- A rigorous and unified mass lumping scheme for higher-order elements is proposed.
- The reason of failure of the common used schemes for mass lumping higher order elements is disclosed.
- The excellent properties of the proposed mass lumping scheme are demonstrated.

Abstract

In dynamic analysis with explicit time integration schemes, a lumped mass matrix (LMM) is preferable, because LMM can avoid solving the large scale simultaneous algebraic equations. Mathematically rigorous mass lumping schemes, such as the mass lumping by nodal quadrature and the row-sum technique, are applicable to only linear or bilinear elements. For higher-order elements, such as 8-node serendipity elements, the diagonal scaling procedure is the only lumping method that can be recommended to generate positive definite diagonal element mass matrices. Unfortunately, there is no mathematical theory in support of this approach. This study proposes a general mass lumping scheme applicable to higher order elements, where the virtual work of initial force is integrated over the problem domain that is viewed as the manifold covered by the finite element patches. By a series of numerical experiments, both free and forced vibration problems, it is suggested that even in the implicit time integration scheme the consistent mass matrix (CMM) can be superseded by the proposed LMM. Furthermore, the proposed LMM has much stronger adaptability to distorted meshes.

Keywords: Lumped mass matrices; High-order elements; Dynamic analysis; Finite element method; Manifolds

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1. Introduction

In structural dynamics applications, lumped mass matrices offer several benefits. Explicit time integration schemes are often necessary in problems such as crash, impact, earthquake and explosion, which require very small time steps to accurately capture the critical physical phenomenon [1]. Furthermore, from a computational viewpoint, implicit time integration algorithms, albeit unconditionally stable, may become uneconomical if large meshes of complex non-linear structures are involved. In explicit time integration algorithms, since there are no numerical iterations at each time step, these algorithms have good properties in terms of accuracy and robustness for problems with strong nonlinearities [2]. With explicit schemes, if a lumped mass matrix (LMM) is employed instead of the consistent mass matrix (CMM), computational cost can be reduced significantly as solving large-scale simultaneous equations associated with the mass matrix is avoided. Even when used in the context of implicit time integration schemes, CMMs have been found to result in spurious oscillations in the solution, and the use of LMMs has been advocated [3]. Therefore, in dynamic analysis, a lumped mass matrix is generally preferable.

Lumped mass matrices have other important applications beyond time integration schemes. The modal method is an important technique to assess the dynamic response of a structure, where a generalized eigenvalue problem is solved. If the mass matrix is diagonal, the generalized eigenvalue problem reduces to a conventional eigenvalue problem, for which powerful solvers are more readily available than for the generalized eigenvalue problem. Moreover, using the same computational effort and memory as for the generalized eigenvalue problem, more modes of the conventional eigenvalue problem can be computed. Such higher frequency modes play an important role in wave propagation analysis [4].

Over the past several decades, the finite element method (FEM) [5] has been the most widely used numerical approach in solving structural dynamics problems. In FEM, the global LMM is obtained by assembling the element LMMs, very similar to forming the global stiffness matrix. An element LMM should satisfy certain constraints, called the admissible conditions [6], such as non-negativity and mass conservation. There have been many mass lumping schemes developed in the finite element literature (see [7–10]), but only the following three schemes appear to be selected in FEM textbooks such as [11–13].

(1) Mass lumping by nodal quadrature [14], where the mass matrix is computed using integration points located at the element nodes.

(2) The row-sum technique [11], in which the lumped mass matrix is obtained by setting

$$M_{ii}^{l} = \sum_{j} M_{ij}^{c}$$

and $M_{ij}^{l} = 0\,(i \neq j)$. Here, M_{ij}^{l} and M_{ij}^{c} represent the components of the element LMM and element CMM, respectively.

(3) The diagonal scaling procedure [3], which sets $M_{ii}^{l} = \alpha M_{ii}^{c}$ and $M_{ij}^{l} = 0\,(i \neq j)$, with the constant α selected to satisfy mass conservation of the element,

$$\sum_{i} M_{ii}^{l} = \int_{\Omega}^{e} \rho dV,$$

with ρ the material density.

In general, the three schemes give different answers. For some higher order elements, such as the 6-node isoparametric triangular element and the 8-node serendipity quadrilateral element, the first two schemes, which have a more definite mathematical basis, give zero or negative diagonal items in the lumped mass matrices, posing significant impediments in practical applications. As a consequence, Hughes said [11]: “Presently, the special lumping technique (called the diagonal scaling procedure in this study and other literature –annotated by the authors) is the only lumping technique that can be recommended for arbitrary elements. Unfortunately, no mathematical theory in support of it has been forthcoming”. And although it has been often used successfully in solid and structural mechanics and heat conduction, a procedure that is not rigorous in theory is always precarious, and indeed some disappointing results have been obtained in fluid mechanics [15].

Apart from the above mentioned schemes, there are also some special mass lumping schemes developed for numerical methods including extended finite element method (XFEM) [16–20], and meshfree method [21–23]. All these procedures are based upon intuition, rather than on a firm mathematical basis.