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Planar domain parameterization for isogeometric analysis based on Teichmüller mapping

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Highlights

- A new approach is proposed for planar domain parameterization based on Teichmüller mapping.
- An effective algorithm is proposed to compute the Teichmüller mapping.
- The new approach guarantees a bijective map and produces more uniform parameterization.
- The new approach demonstrates better numerical property in IGA than other methods.

Abstract

Given the boundary curve of a planar domain, finding a parametric spline representation for the domain is called domain parameterization. A good parameterization of the computational domain plays a key role in isogeometric analysis since it influences the accuracy of the subsequent analysis. In this paper, we propose a new approach for planar domain parameterization based on Teichmüller map—a special map in the class of quasi-conformal map. Under given correspondence of four boundary curves, a unique Teichmüller map between a unit square and a computational domain can be obtained, which guarantees a bijection map and minimizes the maximal conformality distortion. We propose an efficient iterative algorithm to compute the Teichmüller map based on alternating direction method of multipliers (ADMM). Experimental results show that our method can produce more uniform parameterization and increase the accuracy as well as decrease the condition numbers of the stiffness matrices in isogeometric analysis than other state of the art approaches.

Keywords: Parameterization; Isogeometric analysis; Teichmüller mapping; ADMM

1. Introduction

Isogeometric analysis (IGA) provides a new framework for numerical simulation which integrates the two related disciplines of Computer Aided Engineering (CAE) and computer-aided design (CAD) [1,2]. NURBS (non-uniform rational B-splines) are used to represent the geometry of the physical domain and to describe the solution at the same

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time. IGA overcomes unnecessary geometrical conversion from NURBS to meshes in finite element method which is a challenging task, and reduces the computational errors by such conversion. IGA has been successfully applied in many areas, such as linear elasticity, structural vibrations, phase transition phenomena and shape optimization, etc. [3–11].

Given the boundary curve of a computational domain, finding a parametric spline representation for the computational domain is called *domain parameterization*. Domain parameterization greatly influences the computational accuracy in subsequent analysis and thus is an essential step in IGA. The basic requirement for domain parameterization is that the parameterization must be an injective map from the parameterization domain—generally a unit square to the computational domain. Furthermore, the isoparametric curves in the computational domain should be as uniform as possible and isoparametric curves in two orthogonal directions should be as orthogonal as possible. A direct method for domain parameterization is to construct discrete Coons patches introduced by Farin and Hansford [12]. The method is very simple, however the resulting map may not be injective. Gravesen et al. proposed a spring model for domain parameterization, and the quality of resulting parameterization is better than the method based on Coons patches but more expensive [13]. Nguyen et al. applied a sequence of harmonic maps to construct a domain parameterization [14]. The harmonic map is obtained by solving some variational problems, and consequently the injectivity of the map is not guaranteed. Xu et al. put forward a nonlinear optimization method by optimizing some energy function with non-linear constraints such that the parameterization is injective [15]. The method is more expensive since nonlinear optimization problem has to be solved. In [16], Falini et al. proposed three levels of planar domain parameterization techniques based on THB-splines. For complex domains, the domain can be subdivided into subdomains and then each subdomain is separately parameterized such that the whole parameterization is continuous. Such method was described by Xu et al. [17].

For volume parameterization, Aigner et al. proposed a method for swept volumes which cover many free-form shapes in CAD system like blades or propellers [18]. Martin et al. applied discrete volumetric harmonic functions to parameterize a volumetric model and they offer a technique to create a trivariate B-spline that has a consistent parameterization across given isosurfaces [19–21]. In [22,23], the authors extended the nonlinear optimization method proposed in [15] to trivariate multi-block case. Zhang et al. constructed a conformal solid T-spline from boundary T-spline representations for volume parameterization [24].

In this paper, we introduce a special class of quasi-conformal maps called the *Teichmüller maps* for planar domain parameterization. In mathematics, the theory of quasi-conformal maps and Teichmüller maps has a long history and [25] provides a nice introduction about the subject. Quasi-conformal maps can be viewed as a generalization of conformal maps with bounded angle distortion. While conformal maps preserve angles, quasi-conformal maps map an infinitesimal circles to infinitesimal ellipses of bounded eccentricity. A Quasi-conformal map can be characterized by the Beltrami equation with a complex function called *Beltrami coefficient*. The norm of the Beltrami coefficient measures the conformal distortion of the map. A Teichmüller map has the least conformal distortion whose Beltrami coefficient has a constant norm. The theory and applications of quasi-conformal maps are discussed in recent works [26–34], where the problems of constructing quasi-conformal mapping between surfaces, and shape analysis, etc. are considered. Algorithms for computing quasi-conformal maps and Teichmüller maps are fully explored in some recent works, for example, [35–37].

Teichmüller maps have several advantages for domain parameterization problem. First, Teichmüller map ensures a bijection between the parametric domain and the computational domain, which is a necessary requirement in domain parameterization. Second, the Beltrami coefficient of a Teichmüller map has a constant norm, or in other words, Teichmüller map has uniform conformality distortion over the computational domain, which ensures that the isoparametric curves in the computational domain are as uniform as possible. Thirdly, a Teichmüller map is an extremal quasi-conformal map (see [38] for example) which ensures that isoparametric curve networks are as orthogonal as possible.

The remainder of the paper is organized as follows. Section 2 presents the background and the preliminary knowledge about quasi-conformal maps and Teichmüller maps. Section 3 describes a detailed algorithm to compute a Teichmüller map from a unit square to a computational domain based on B-spline representations. Experimental results and comparisons are presented in Section 4. Finally, we give a conclusion of this paper and future works in Section 5.

2. Background and preliminary knowledge

In this paper, we will use complex coordinates in the plane. Let $z = x + \mathrm{i}y$ represent a complex variable with x and y being the real part and imaginary part of z respectively, and $\bar{z} = x - \mathrm{i}y$ is the complex conjugate of z. For an arbitrary