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Topological sensitivity analysis in heterogeneous anisotropic elasticity problem. Theoretical and computational aspects

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Highlights

- Topological derivative of elastic anisotropic and heterogeneous 2D problem.
- Novel, extremely simple closed formula for the topological sensitivity.
- Full mathematical justifications for the obtained formulas.
- Potential applications to structural topology design.

Abstract

The topological sensitivity analysis for the heterogeneous and anisotropic elasticity problem in two-dimensions is performed in this work. The main result of the paper is an analytical closed-form of the topological derivative for the total potential energy of the problem. This derivative displays the sensitivity of the cost functional (the energy in this case) when a small singular perturbation is introduced in an arbitrary point of the domain. In this case, we consider a small disc with a completely different elastic material. Full mathematical justification for the derived formula, and derivations of precise estimates for the remainders of the topological asymptotic expansion are provided. Finally, the influence of the heterogeneity and anisotropy is shown through some numerical examples of structural topology optimization.

Keywords: Topological derivative; Structural optimization; Topology design; Elastic anisotropy

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1. Introduction

Topological asymptotic analysis allows obtaining an asymptotic expansion of a given shape functional when a geometrical domain is singularly perturbed. This perturbation can be materialized by the insertion of holes, inclusions, source-terms or even cracks. The main concept arising from this analysis is the topological derivative [1]. This derivative measures the sensitivity of the shape functional with respect to the infinitesimal singular domain perturbation and it was rigorously introduced in [2]. Since then, this concept has proven extremely useful in the treatment of a wide range of problems; see, for instance, [3–11]. Concerning the theoretical development of the topological asymptotic analysis, besides the monograph [1], the reader is referred to [12,13].

In order to introduce these concepts, let us consider an open and bounded domain $\Omega \subset \mathbb{R}^2$, see Fig. 1, which is subject to a non-smooth perturbation confined in a small region $\omega_\varepsilon(\widehat{x}) = \widehat{x} + \varepsilon\omega$ of size ε. Here, $\widehat{x}$ is an arbitrary point of Ω and ω is a fixed domain of $\mathbb{R}^2$. Then, we assume that a given shape functional $\mathcal{J}_\varepsilon(\Omega)$, associated to the topologically perturbed domain, admits the following topological asymptotic expansion [1]

$$\mathcal{J}_\varepsilon(\Omega) = \mathcal{J}(\Omega) + f(\varepsilon)\mathcal{T}(\widehat{x}) + o(f(\varepsilon)), \tag{1}$$

where $\mathcal{J}(\Omega)$ is the shape functional associated to the unperturbed domain and $f(\varepsilon)$ is a positive function such that $f(\varepsilon) \to 0$ when $\varepsilon \to 0^+$. The function $\widehat{x} \mapsto \mathcal{T}(\widehat{x})$ is termed the topological derivative of $\mathcal{J}$ at $\widehat{x}$. Therefore, the term $f(\varepsilon)\mathcal{T}(\widehat{x})$ represents a first order correction of $\mathcal{J}(\Omega)$ to approximate $\mathcal{J}_\varepsilon(\Omega)$ in $\widehat{x}$. In this work, the singular perturbation is characterized by a circular disc, denoted B_ε, with boundary ∂B_ε and different constitutive properties, see Fig. 1.

From (1), we obtain the standard definition of the topological derivative by passing to the limit $\varepsilon \to 0^+$:

$$\mathcal{T}(\widehat{x}) = \lim_{\varepsilon \to 0^+} \frac{\mathcal{J}_\varepsilon(\Omega) - \mathcal{J}(\Omega)}{f(\varepsilon)}. \tag{2}$$

Notice that, since we are dealing with singular domain perturbations, the shape functionals $\mathcal{J}_\varepsilon(\Omega)$ and $\mathcal{J}(\Omega)$ are associated to topologically different domains. Therefore, the above limit is not trivial to be calculated. In particular, we need to perform an asymptotic analysis of the shape functional $\mathcal{J}_\varepsilon(\Omega)$ with respect to the small parameter ε, i.e. we need information of $\mathcal{J}_\varepsilon(\Omega)$ when $\varepsilon \to 0^+$. As it will be shown later, the shape functional difference (2) depends on the polarization tensor, which is considered a fundamental concept on the topological derivative topic. This tensor, also known in the literature as *Pólya–Szegö polarization tensor*, arises from the asymptotic analysis in singularly perturbed geometrical domains [14]. This mathematical concept permits to write an asymptotic expansion of the shape functional $\mathcal{J}_\varepsilon(\Omega)$ by means of functions evaluated in the unperturbed domain Ω (without considering B_ε). The polarization tensor is characterized by a matrix – *polarization matrix* – depending only on the constitutive properties of the problem and the shape of the singular domain perturbation [15].

The topological derivative, in its closed form, has been fully developed for a wide range of physical phenomena. Most of them, by considering homogeneous and isotropic constitutive behaviors. In fact, only a few works dealing with heterogeneous and anisotropic material behavior can be found in the literature, and, in general, the derived formulas are given in an abstract form (see, for instance, [12]). Closed and analytical forms for this kind of constitutive behavior have been only developed for heat diffusion problems (see [16–18,2]). For anisotropic elasticity, the existence and properties of the polarization tensor was studied in [19,20]. However, the polarization tensor is given again in an abstract form. A technique for the numerical evaluation of the polarization tensor is presented in [21].

In this work, we derive the topological derivative in its closed form for the total potential energy associated to an anisotropic and heterogeneous elasticity problem. We assume as singular perturbation a small circular inclusion introduced at an arbitrary point of the domain. The constitutive properties of the small disc are also anisotropic and completely different from the elasticity properties of the matrix. In addition, we provide a full mathematical justification of the derived formula, and develop precise estimates for the remainders of the topological asymptotic expansion.

Bearing this in mind, the heterogeneous anisotropic topological derivative concept, can be applied in advanced technological research areas such as topology and structural optimization simultaneously combined with topological material-design. In fact, in multi-scale modeling, for a given microstructure the homogenized constitutive response is, in general, anisotropic. In addition, since in each macroscopical structural point we have a different microstructure, the constitutive homogenized response at the macro-scale varies from point to point, i.e., it is heterogeneous. Therefore,