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Pressure-robustness and discrete Helmholtz projectors in mixed finite element methods for the incompressible Navier–Stokes equations

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Highlights

- Novel pressure-robust Stokes discretizations, whose velocity errors are pressure-independent, are applied to the steady Navier–Stokes equations.
- A novel, pressure-robust treatment of the nonlinear convection term leads to large speedups for potential flows.
- The notion of a discrete Helmholtz projector of an inf–sup stable Stokes mixed method is introduced.
- The concept of a discrete Helmholtz projector is applied in the numerical analysis of the Stokes and Navier–Stokes equations.

Abstract

Recently, it was understood how to repair a certain L^2-orthogonality of discretely divergence-free vector fields and gradient fields such that the velocity error of inf–sup stable discretizations for the incompressible Stokes equations becomes pressure-independent. These new ‘pressure-robust’ Stokes discretizations deliver a small velocity error, whenever the continuous velocity field can be well approximated on a given grid. On the contrary, classical inf–sup stable Stokes discretizations can guarantee a small velocity error only when both the velocity and the pressure field can be approximated well, simultaneously.

In this contribution, ‘pressure-robustness’ is extended to the time-dependent Navier–Stokes equations. In particular, steady and time-dependent potential flows are shown to build an entire class of benchmarks, where pressure-robust discretizations can outperform classical approaches significantly. Speedups will be explained by a new theoretical concept, the ‘discrete Helmholtz projector’ of an inf–sup stable discretization. Moreover, different discrete nonlinear convection terms are discussed, and skew-symmetric pressure-robust discretizations are proposed.

Keywords: Incompressible Navier–Stokes equations; Mixed finite element methods; A-priori error estimates; Pressure-robustness; Helmholtz projector; Irrotational forces

1. Introduction

Though inf–sup stable mixed finite elements for the incompressible Stokes equations [1–3] are a seemingly mature research field, the concept of a *pressure-robust* mixed method [4] was introduced only recently. A pressure-robust

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mixed method for the incompressible Stokes equations

$$\begin{aligned} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f}, \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

– with velocity field $\mathbf{u}$, pressure p, an exterior forcing $\mathbf{f}$ and a kinematic viscosity $\nu > 0$ – denotes a convergent discretization, whose velocity error *does not depend on the continuous pressure*. In fact, the velocity errors of classical inf–sup stable mixed methods like the nonconforming Crouzeix–Raviart element or the H^1-conforming Taylor–Hood element depend on a possibly large (for small ν) pressure-dependent error contribution [5]

$$\frac{1}{\nu} \inf_{q_h \in Q_h} \|p - q_h\|_{L^2},$$

where Q_h denotes the discrete pressure space. Therefore, classical inf–sup stable mixed finite elements are optimally convergent, but only deliver a useful velocity error in the case $\nu = \mathcal{O}(1)$, when the continuous velocity $\mathbf{u}$ and the pressure p are simultaneously well-resolved on a given grid. In the literature, this lack of robustness is sometimes called *poor mass conservation* [6,7], and is traditionally mitigated by *grad–div stabilization* [5,8–12]. *Pressure-robust* schemes instead, deliver a velocity error, which is independent of the continuous pressure p (and of the size of the kinematic viscosity ν). Only the velocity has to be resolved well, in order to deliver a small velocity error. The first inf–sup stable and pressure-robust mixed method on (barycentrically refined) unstructured meshes in 3D was proposed by S. Zhang [13], which is a 3D analogon of a 2D element analyzed by J. Qin [14]. Later on the Scott–Vogelius element on unstructured barycentrically refined meshes was investigated with respect to practical applicability in several works [15,16,6]. Since then, several new and interesting pressure-robust, inf–sup stable Stokes elements have been constructed, see for example the recent works [17–19] and further references in [20].

This contribution now applies *pressure-robust* finite element Stokes discretizations to the time-dependent incompressible Navier–Stokes equations

$$\begin{aligned} \mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{f}, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned}$$

and elaborates on the question, when these schemes outperform classical inf–sup stable discretizations. In order to answer this question convincingly, two inf–sup stable discretizations are investigated: the first order Bernardi–Raugel element and the second order P_2^+-P_1^{disc} element. These discretizations are compared to some recent pressure-robust variants [21,20], which have the same degrees of freedom for velocity and pressure. The corresponding pressure-robust variants are constructed in the Stokes case by modifying the L^2 scalar product in the discretization of the exterior force $\mathbf{f}$ by

$$\int_D \mathbf{f} \cdot \mathbf{v}_h \, dx \to \int_D \mathbf{f} \cdot \Pi \mathbf{v}_h \, dx, \tag{1.1}$$

which repairs a certain L^2 orthogonality between discretely divergence-free vector fields and gradient fields. Here, $\Pi \mathbf{v}_h$ denotes an $\mathcal{O}(h)$ approximation of $\mathbf{v}_h$, whose *weak divergence* $\nabla \cdot \Pi \mathbf{v}_h$ coincides with the *discrete divergence* of $\mathbf{v}_h$. In the time-dependent incompressible Navier–Stokes case, this contribution shows that similar modifications of the L^2 scalar products in the nonlinear convection term $(\mathbf{u}_h \cdot \nabla)\mathbf{u}_h$, and the (approximative) time derivative $\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\mathrm{dt}}$ can lead to remarkably more accurate velocity approximations. This will be practically demonstrated focusing on a classical class of benchmark flows: potential flows. For potential flows $\mathbf{u} = \nabla \chi$ with χ being a (maybe time-dependent) harmonic potential, the nonlinear convection term and the time derivative are gradient fields: $(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{2}\nabla(|\mathbf{u}|^2)$, $\mathbf{u}_t = \nabla(\chi_t)$. Therefore, they are balanced by the pressure gradient ∇p in the momentum balance, which makes the pressure p comparably large in these benchmarks, though the exterior force equals $\mathbf{f} = \mathbf{0}$. In a nutshell, potential flows show that *pressure-robust schemes* merit their name also in the time-dependent Navier–Stokes problem, — since they outperform classical inf–sup stable schemes due to large pressures.

Mathematically, this reasoning can be made more precise by looking at the Helmholtz projector $\mathbb{P}(\mathbf{f})$ of a vector field $\mathbf{f} \in L^2(D)^m$, which denotes its divergence-free part in the sense of the Helmholtz decomposition. For potential flows, the identities $\mathbb{P}((\mathbf{u} \cdot \nabla)\mathbf{u}) = \mathbf{0}$ and $\mathbb{P}(\mathbf{u}_t) = \mathbf{0}$ hold, since the time derivative and the convection term are