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High-order central difference scheme for Caputo fractional derivative

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Abstract

In this paper we propose a class of central difference schemes for resolving the Caputo fractional derivative. The accuracy may reach any selected integer order. More precisely, the Caputo fractional derivative operator is decomposed into symmetric and antisymmetric components. Starting from difference schemes of lower order accuracy for each component, we enhance the accuracy by a weighted average of shifted differences. The weights are calculated by matching the symbols of the scheme and the operators. We further illustrate the application of the proposed schemes to a fractional advection–diffusion equation. Together with the Crank–Nicolson algorithm, it reaches designed accuracy order, and is unconditionally stable. Numerical tests are presented to demonstrate the nice features.

Keywords: Caputo fractional derivative; Fractional calculus; Central difference scheme; Advection–diffusion equation

1. Introduction

In recent two decades, fractional partial differential equations (FPDE) have caught a lot of attentions due to their inherent ability for modelling non-local phenomena in various fields of science and engineering [1,2]. Meanwhile, the non-local feature brings about many challenges to the existing numerical methods in terms of memory storage, computational cost, accuracy, etc. In this paper, we propose a class of high order central difference schemes, and adopt them to construct unconditionally stable finite difference algorithm for solving a fractional advection–diffusion equation (FADE) as follows, which was proposed to describe anomalous diffusion [3].

$$\begin{cases} \partial_t \phi(x,t) = \kappa_1 \, {}_0^C D_x^\alpha \phi(x,t) + \kappa_2 \, {}_x^C D_L^\alpha \phi(x,t) - u \partial_x \phi(x,t) + g(x,t), \\ \phi(0,t) = 0; \; \phi(L,t) = 0; \\ \phi(x,0) = \phi_0(x), \end{cases} \tag{1}$$

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where $\kappa_1 \geq 0, \kappa_2 \geq 0$ are diffusion coefficients, and u is an advection velocity. The left-sided and right-sided Caputo fractional derivative operators ${}_0^C D_x^\alpha$ and ${}_x^C D_L^\alpha$ of a function $\phi(x)$ are defined for $1 < \alpha < 2$ as follows.

$$ {}_0^C D_x^\alpha \phi(x) = \frac{1}{\Gamma(2-\alpha)} \int_0^x (x-\eta)^{1-\alpha} \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} \phi(\eta) d\eta, \tag{2} $$

$$ {}_x^C D_L^\alpha \phi(x) = \frac{1}{\Gamma(2-\alpha)} \int_x^L (\eta-x)^{1-\alpha} \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} \phi(\eta) d\eta. \tag{3} $$

Being one of the dominant numerical tools in scientific computing and engineering applications, finite difference method (FDM) is naturally adopted to solve fractional differential equations (FDE) for its simplicity to apply. For the two most widely used fractional derivatives, the Caputo and Riemann–Liouville fractional derivatives, there exist mainly three types of FDM classified by how they are derived. The basic idea of the first type of FDM is direct numerical approximation to the expression of fractional derivatives. Among this type of methods, a certain kind of schemes called *L1 formula* can achieve a numerical accuracy of order $O(h^{2-\alpha})$ [4–7], where $0 < \alpha < 1$ is the fractional order and h stands for the grid size. This formula is established by a piecewise linear interpolation approximation to the integrand function. To improve the numerical accuracy, Gao [8] discretized the Caputo fractional derivative with a numerical accuracy of $O(h^{3-\alpha})$ by constructing a quadratic interpolation for the integrand function. Then Alikhanov [9] improved Gao's scheme to prove the stability and convergence. More similar works achieving higher order up to $O(h^{6-\alpha})$ can be found in [10,11]. The second type of FDM is obtained by means of the Grünwald–Letnikov fractional derivative, which is defined as a finite difference. Igor [12] used it as a numerical approximation for the Riemann–Liouville fractional derivative with an accuracy order of $O(h)$. To improve the accuracy order and achieve stability for solving related differential equations, the Grünwald–Letnikov fractional derivative is shifted and summed with some weights, which leads to a class of second order discretizations [13,14]. The third type of FDM is based on the fundamental work by Lubich [15], who introduced fractional linear multi-step methods to approximate fractional integrals. Later on, Zeng [16] solved a time-fractional diffusion equation with unconditional stability, where the Caputo fractional derivative was approximated by linear multi-step methods with second order accuracy. Chen [17] derived a class of fourth order approximations for the Riemann–Liouville derivative by the same methods. More finite difference schemes for fractional derivatives have been discussed in Li's recent book [18].

In this paper, a new type of high-order central difference schemes for Caputo fractional derivative are proposed. To this end, Caputo fractional derivative is decomposed into a symmetric operator and an antisymmetric one. The symmetric part is actually related to Riesz fractional derivative up to a constant. Based on some existing low order finite difference schemes, a class of central difference schemes are designed as a weighted average of some shifted differences. The weights are obtained by matching the symbols of the schemes and the operators. We remark that these symbols are also referred to as the generating functions of finite difference schemes for Riesz fractional derivative [19–21]. Such designs can reach any desired order of accuracy, up to infinity which means exact resolution. We present the weights for finite difference schemes of up to $O(h^8)$ as examples. The 'infinite-order' finite difference for Caputo derivative is also discussed. After this, by using the Crank–Nicolson algorithm for time integration, we present a numerical algorithm to solve (1). We also show the unconditional stability by the von Neumann analysis, and display the convergence rate of the algorithm. Algorithms of order $O(h^p)$, where $p = 2, 4, 6, 8, +\infty$, are shown as examples.

Compared with the existing FDM in the literature, the new schemes have the following features:

- The Caputo fractional derivatives of any order $\alpha > 0$ can be approximated at accuracy of any desired even integer, up to infinite order.
- The accuracy order of the scheme is not affected by the order of the fractional derivative, which is different from the aforementioned first type of FDM. We remark that the mechanism is similar to the Moving Least Square Reproducing Kernel interpolation [22].
- The structure of the scheme is symmetric, with all information on both sides of the reference point being used.

The rest of this paper is organized as follows. In Section 2, Caputo fractional derivative is decomposed. Then the schemes for each component are obtained by matching symbols, leading to high-order schemes for Caputo fractional derivative. In Section 3, the proposed high-order schemes are used to solve (1), where unconditional stability and convergence of the algorithm are proved. In Section 4, two numerical examples are presented. Finally, the paper is concluded in Section 5.