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An admissibility and asymptotic preserving scheme for systems of conservation laws with source term on 2D unstructured meshes with high-order MOOD reconstruction

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Highlights

- A high-order AP finite volumes scheme is designed for 1D and 2D unstructured meshes.
- The MOOD paradigm is used to preserve the set of admissible states.
- Convergence towards the diffusion limit is studied by inspecting rates of convergence.
- Reference solutions with source terms are constructed to compare the results.

Abstract

The aim of this work is to design an explicit finite volume scheme with high-order MOOD reconstruction for specific systems of conservation laws with stiff source terms which degenerate into diffusion equations. We propose a general framework to design an asymptotic preserving scheme that is stable and consistent under a classical hyperbolic CFL condition in both hyperbolic and diffusive regimes for any 2D unstructured mesh. Moreover, the developed scheme also preserves the set of admissible states, which is mandatory to conserve physical solutions in stiff configurations. This construction is achieved by using a non-linear scheme as a target scheme for the limit diffusion equation, which gives the form of the global scheme for the full system. The high-order polynomial reconstructions allow to improve the accuracy of the scheme without getting a full high-order scheme. Numerical results are provided to validate the scheme in every regime.

Keywords: Asymptotic-preserving schemes; Finite volumes schemes; Hyperbolic systems of conservation laws with source terms; MOOD

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1. Introduction

In this work we study the numerical approximation of systems of conservation laws with stiff source terms, which can be written in the following formalism:

$$\partial_t \mathbf{U} + \operatorname{div}(\mathbf{F}(\mathbf{U})) = \gamma(\mathbf{U})(\mathbf{R}(\mathbf{U}) - \mathbf{U}), \tag{1.1}$$

where the vector of conservative variable $\mathbf{U}$ is in $\mathcal{A} \subset \mathbb{R}^n$ the set of admissible states. The source term is composed of γ a positive function that controls its stiffness and $\mathbf{R}(\mathbf{U}) \in \mathcal{A}$. The homogeneous hyperbolic system associated to (1.1) is:

$$\partial_t \mathbf{U} + \operatorname{div}(\mathbf{F}(\mathbf{U})) = 0, \tag{1.2}$$

with $\mathbf{F}$ the hyperbolic physical flux. The compatibility conditions from [1] are assumed to be fulfilled so that when $\gamma(\mathbf{U})t \to \infty$ the system (1.1) degenerates into a diffusion equation:

$$\partial_t u - \operatorname{div}\big(D(u)\nabla u\big) = 0, \tag{1.3}$$

where $u \in \mathbb{R}$ is linked to $\mathbf{U}$, and D is positive function.

The key point of this work is to construct high-order schemes which extend the 1D asymptotic preserving (AP for short) scheme designed in [2] and the one for 2D unstructured meshes from [3]. This extension to high-order has to be done without losing the AP property of the first-order schemes.

The AP property will be considered in the sense of Jin [4] that is Fig. 1 holds with uniform bounds on the parameters in terms of γt (e.g. the CFL condition).

Throughout this paper, the isentropic Euler model with friction is used as an example:

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho\mathbf{u} \end{pmatrix}, \mathbf{F}(\mathbf{U}) = \begin{pmatrix} (\rho\mathbf{u})^T \\ \rho\mathbf{u} \otimes \mathbf{u} + p\mathbf{I} \end{pmatrix}, \mathbf{R}(\mathbf{U}) = \begin{pmatrix} \rho \\ 0 \end{pmatrix}, \tag{1.4}$$

where $\gamma(\mathbf{U}) = \kappa(\rho) > 0$ is the friction coefficient and $p(\rho)$ a pressure law with $p'(\rho) > 0$. For all the test cases we choose $p(\rho) = \rho^{1.4}$. The set of admissible states of this model is, in 2D:

$$\mathcal{A} = \{\mathbf{U} = (\rho, \rho\mathbf{u})^T \in \mathbb{R}^3/\rho > 0\}.$$

When $\kappa t \to \infty$, the system degenerates into the following diffusion equation (see [5–7,1] for a rigorous proof):

$$\partial_t \rho - \operatorname{div}\left(\frac{p'(\rho)}{\kappa(\rho)}\nabla\rho\right) = 0. \tag{1.5}$$

Various others systems enter the framework of (1.1), including the P_1 model for radiative transfer [8] or the M_1 model [9,2,3]. Let us emphasize the fact that γ does indeed strongly depends on $\mathbf{U}$ in several applications. Hence, it is crucial to build a numerical scheme able to deal with this feature.

There is a strong interest in developing AP schemes in our context since the pioneer work of Gosse and Toscani [10]. This work has been generalized in [2] for all systems in our formalism (1.1) for the 1D case. Then, several works have been done to construct AP schemes in 2D, by using 1D techniques [11,12,3] or MPFA based schemes [13,8,14,15]. The aim is now to develop high-order finite volume schemes for those systems, especially for 2D unstructured meshes. IMEX methods [16] and (W)ENO discretizations [17,18] have been used to get up to second and third-order schemes in [19,20] for some 1D hyperbolic systems and kinetic equations. When dealing with kinetic equations one may also use projective integrations [21] or again IMEX time schemes with micro–macro decomposition and Discontinuous Galerkin [22] to construct high-order schemes, for instance. This list of techniques is not exhaustive and for more details about each method the reader is referred to the cited works and references therein.

The content of this work is divided into two main parts: one for 1D schemes and another one for 2D schemes conducted on unstructured meshes. Inside each part, the first-order scheme is quickly recalled for the homogeneous hyperbolic system (1.2) then extended to high-order. Then, the construction of a scheme using high-order polynomial reconstruction of the MOOD method [23] is also done with the two AP schemes for the system with source term (1.1) in 1D and 2D. Besides, results are presented with reference solutions constructed in the appendices, and the convergence towards the diffusion limit is investigated.