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Blending low-order stabilised finite element methods: A positivity-preserving local projection method for the convection–diffusion equation

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Abstract

In this work we propose a nonlinear blending of two low-order stabilisation mechanisms for the convection–diffusion equation. The motivation for this approach is to preserve monotonicity without sacrificing accuracy for smooth solutions. The approach is to blend a first-order artificial diffusion method, which will be active only in the vicinity of layers and extrema, with an optimal order local projection stabilisation method that will be active on the smooth regions of the solution. We prove existence of discrete solutions, as well as convergence, under appropriate assumptions on the nonlinear terms, and on the exact solution. Numerical examples show that the discrete solution produced by this method remains within the bounds given by the continuous maximum principle, while the layers are not smeared significantly.

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1. Introduction

The design and analysis of stabilised finite element methods for convection–diffusion equations remains a challenging problem. In particular if the method is required to have (close to) optimal convergence where the exact solution is smooth, but to preserve the monotonicity properties of the continuous problem in the vicinity of layers. The standard approach has been to combine a linear stabilisation method that ensures accuracy in the smooth part of the solution and control of the propagation of perturbations from layers with a nonlinear so-called shock-capturing term

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that is designed to diminish, or ideally, eliminate the local spurious oscillations close to the layer. For an overview and a critical evaluation of such methods we refer to [1,2] and references therein.

The design of such shock-capturing terms has typically been residual based [3–5], matching residual based stabilisation methods such as SUPG. The idea that finite element shock-capturing terms should be designed with the objective of satisfying a discrete maximum principle was pioneered by Mizukami and Hughes in [6], and further discussed in [4,5]. However when symmetric stabilisation methods such as local projection stabilisation (LPS) or continuous interior penalty (CIP) methods are used, classical shock-capturing terms appear to be less natural. Instead, our objective in this paper is to explore the idea of designing a method that switches from a low order, but monotone, method that acts in the vicinity of layers, to an optimal, non monotone, method that is active in smooth regions. This main philosophy can be tracked back to the seminal work [7], and has been explored in different guises since, especially in the context of algebraic flux correction schemes in [8–11], and more recently in the work of Kuzmin et al. [12–17], and Guermond et al. [18,19]. See also the recent work [20] for an idea based on a low-order local-projection type method, applied to the transport problem.

In this work we propose a particular realisation of the general approach described in the previous paragraph. More precisely, we will develop an idea introduced in [21]. Therein it was suggested that in the framework of a local projection method (or subgrid viscosity method), where the stabilisation takes the form of a penalty on the gradient of u_h minus the projection of u_h onto some smaller space, i.e. on $\nabla(u_h - \pi_H u_h)$, a nonlinear switch $\alpha(u_h) \in [0, 1]$ could be introduced, $\nabla(u_h - \alpha(u_h)\pi_H u_h)$ taking the value 1 in the smooth part and 0 close to layers, hence turning off the projection part in the vicinity of layers. This makes the stabilisation degenerate to first order viscosity in the non-smooth part of the approximate solution so that the spurious oscillations are damped or even completely eliminated provided the mesh satisfies certain geometric conditions.

It turns out, however, that since the parameters for the first order viscosity and the LPS-term are of different magnitude, the idea cannot be realised in this simple fashion, but instead the first order linear diffusion and the high order stabilisation term must be blended together locally using the nonlinear switch $\alpha(u_h)$ (similar approaches have been advocated recently by Ern and Guermond [22] and Badia and Hierro [23], using different stabilisation methods and slightly different focus). Below we design a nonlinear LPS method based on these ideas. We show that the method satisfies a discrete maximum principle under suitable assumptions on the mesh (depending on the diffusion operator), that the nonlinear discrete problem admits (at least) one solution and discuss what properties are required from the approximate solution and the nonlinear stabilisation in order to obtain an optimal a priori error estimate for smooth solutions, including the effect both of the linear and the nonlinear stabilisation operator. The above results are, essentially, independent of the concrete definition of the blending parameter, as long as it satisfies the basic requirements. We modify slightly two known limiters that have been applied in the context of Algebraic Flux Correction (AFC) schemes, and use them as a blending parameter. We then test them numerically, focusing on the accuracy and elimination of spurious oscillations.

The rest of the manuscript is organised as follows. The remainder of this introduction will be devoted to present the notations and necessary preliminary results. The bulk of this work is Section 2, where we describe the linear diffusion and LPS methods used in this work, and the way to blend them. An existence and convergence analysis is carried out, and the discrete maximum principle is discussed, under rather general assumptions on the nonlinear switch $\alpha(u_h)$. The two definitions used for this switch are presented in Section 3, and are tested by several numerical experiments in Section 4. Finally, we draw some conclusions and perspectives.

1.1. The model problem, notations and preliminary results

Throughout this work we adopt standard notation for Sobolev spaces. In particular, for $D \subset \mathbb{R}^d$ we denote by $(\cdot,\cdot)_D$ the inner product in $L^2(D)$ (or $L^2(D)^d$, if necessary). For $\ell \geq 0$, we denote by $\|\cdot\|_{\ell,D}$ ($|\cdot|_{\ell,D}$) the norm (seminorm) in $H^\ell(D)$. We will also adopt the usual convention that $H^0(D) = L^2(D)$.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded polygonal (polyhedral) domain with a Lipschitz-continuous boundary $\partial\Omega$. We consider the steady-state convection–diffusion–reaction equation

$$\begin{aligned} -\varepsilon\,\Delta u + \boldsymbol{b}\cdot\nabla u + \sigma\,u &= f \quad \text{in } \Omega\,, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$