Available online at www.sciencedirect.com
ScienceDirect
Computer methods in applied mechanics and engineering
Comput. Methods Appl. Mech. Engrg. 315 (2016) 169–189
www.elsevier.com/locate/cma

A stabilized mixed finite element method for coupled Stokes and Darcy flows with transport☆

Hongxing Rui*, Jingyuan Zhang

School of Mathematics, Shandong University, Jinan, 250100, China

Received 11 August 2016; received in revised form 24 October 2016; accepted 24 October 2016
Available online 3 November 2016

Highlights

- The model is with the viscosity depends on the solute concentration.
- The velocity, pressure and concentration are approximated, respectively.
- The existence, uniqueness and optimal order a priori error estimates are derived.
- No restriction of the dependence of time-step on spatial meshsize is needed.

Abstract

In this article, we present a stabilized mixed finite element method for solving the coupled Stokes and Darcy flow equations with a solute transport. The mathematical model includes the velocity and pressure equations and concentration equation where the viscosity depends on the concentration. We propose a mixed weak formulation and use the nonconforming piecewise Crouzeix–Raviart finite element, piecewise constant and conforming piecewise linear finite element to approximate velocity, pressure and concentration respectively. The existence, uniqueness of the approximate solution are obtained, and optimal order a priori error estimates are derived. No assumption on the boundness of the infinity norms of approximate velocity or concentration or the restriction about the time-step and spatial meshsize is needed due to a new weak formulation introduced for the concentration equation. Numerical examples are presented to verify the theoretical results.
Published by Elsevier B.V.

MSC: 65M12; 65M15; 65M60

Keywords: Coupled Stokes and Darcy flows; Solute transport; Nonconforming Crouzeix–Raviart element; Mixed element and finite element; Error estimates

1. Introduction

Recent years, there is an intense research in building suitable mathematical and numerical models for the fluid movement which flows across a porous medium and a free fluid region. The models have significance and wide

☆ The work is supported by the National Natural Science Foundation of China Grant No. 91330106, 11671233.
* Corresponding author.
E-mail address: hxrui@sdu.edu.cn (H. Rui).

http://dx.doi.org/10.1016/j.cma.2016.10.034
0045-7825/Published by Elsevier B.V.

applications in hydrology, environment science and biofluid dynamics. For example, the models appear in the research of karst aquifers [1] or in filter design [2].

There are many papers to develop the numerical methods for the coupled Stokes and Darcy flow problems. Many kinds of numerical methods have been studied deeply, for example, the domain decomposition methods [3–7], the finite element methods [8–10], the non-conforming finite element methods [11], the Mortar multiscale finite element methods [12], the Lagrange multiplier and mixed element methods [13–17], the mixed finite element method combining with the DG method [18,19], the DG method combining with mimetic finite difference method [20], the pseudospectral least squares method [21] and spectral method [22], the partitioned time stepping algorithm [23], and many other numerical methods [24–30]. All these papers only consider the single component flow problems where just velocity and pressure appear in the models.

However, the models describing solute transport in the coupled fluid flow region and porous media flow region appear in a variety of physical phenomena that affect the human health and the environment at a large scale. For instance, the pollution of groundwater by transport of contaminants through rivers and lakes is an important environmental problem. The enhanced oil recovery by injecting chemical component into the Karst aquifer including vuggy region is an energy problem with solute transport. For this model, Cesmelioglu and Rivière [31] study the existence and stability bounds of the weak solution with the fluid viscosity depending on the concentration.

Within our knowledge, there is only two papers [32,33] to study the numerical method for the coupled Stokes–Darcy flow with transport, where the stationary Stokes equation is used. The flow equations for velocity and pressure are solved using finite element, mixed element method or discontinuous Galerkin method, and the concentration equation is solved using the discontinuous Galerkin methods. But in this paper the viscosity of the fluid was assumed to be independent of the concentration, this assumption decouples the flow equations from the concentration equation. Up to now there is no paper to consider the numerical methods for the models with viscosity depending on concentration, even though they are more accurate, realistic and important.

The purpose of this paper is to study the numerical methods for the coupled Stokes and Darcy problem with solute transport. The mathematical model includes the coupled Stokes and Darcy equation with an advection–diffusion equation modeling the transport of a chemical component, where the viscosity is concentration-dependent and the Stokes equation is non-stationary. It is a strongly coupled nonlinear system including pressure, velocity and concentration. We construct a stabilized mixed finite element method for the coupled flow problem by using the nonconforming piecewise linear Crouzeix–Raviart element for the velocity and piecewise constant function for pressure. And we use classical piecewise linear finite element for concentration.

The usual method to analyze the stability and evaluate the a priori errors is to use the assumption that the infinitive norms of approximate velocity and concentration are bounded, which will introduce some restrictions on the discretization parameters and finite element spaces, for example, it requires that the index of finite element spaces should be greater than or equal to 2. In this paper since we just use the piecewise finite element spaces, the usual existed methods cannot be used. To overcome the difficulty we introduce a new formulation $d(\boldsymbol{u}; \cdot, \cdot)$ for the concentration equation, which is proved to be positive definite, see Lemma 3.7. By using this $d(\boldsymbol{u}; \cdot, \cdot)$, no assumption on the boundness of infinitive norms of approximate velocity or concentration or the restriction about the time-step and spatial meshsize is needed in error estimates, see the analysis in Section 4. We prove that the approximate solution of the system exists uniquely and obtain the optimal error estimates for velocity and concentration. The introduced $d(\boldsymbol{u}; \cdot, \cdot)$ can also be used to construct and analyze other kinds of numerical methods.

After then some numerical examples are carried out to verify that the numerical results are in agreement with the theoretical analysis.

The rest of the article is organized as follows. In Section 2, we introduce the model problem, the assumptions on the physical data and present the mixed weak formulation. In Section 3, we propose the fully discrete finite element formulation with a stabilization term added to derive the discrete inf–sup condition. The error estimates for the velocity, pressure and concentration in L^2 norm are presented in Section 4. In Section 5, we present some numerical examples.

Throughout this paper we use K, with or without subscription, to denote a generic constant, which should have different values in different appearances.

2. Model problem and weak formulation

The model we consider is a flow in a bounded two dimensional domain $\Omega \subset R^N$ ($N = 2$ or 3), consisting of a fluid flow region Ω_s, where the flow is governed by the Stokes equation, and a porous medium region $\Omega_d = \Omega \setminus \overline{\Omega}_s$,