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# Postprocessing of non-conservative flux for compatibility with transport in heterogeneous media

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### Highlights

- Present an order-preserving algorithm to postprocess non-conservative fluxes on a wide range of grids.
- Add a piecewise constant correction term that is minimized in a weighted $L^2$ norm.
- Application of a weighted norm appears to give better results for high contrasts in permeability.
- Study both steady-state and dynamic flow models.
- Solve coupled flow and transport problem to demonstrate effect of postprocessing.

### Abstract

A conservative flux postprocessing algorithm is presented for both steady-state and dynamic flow models. The postprocessed flux is shown to have the same convergence order as the original flux. An arbitrary flux approximation is projected into a conservative subspace by adding a piecewise constant correction that is minimized in a weighted $L^2$ norm. The application of a weighted norm appears to yield better results for heterogeneous media than the standard $L^2$ norm which has been considered in earlier works. We also study the effect of different flux calculations on the domain boundary. In particular we consider the continuous Galerkin finite element method for solving Darcy flow and couple it with a discontinuous Galerkin finite element method for an advective transport problem.



*Keywords:* Postprocessing; Local conservation; Galerkin FEM; Darcy flow; Advective transport

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## 1. Introduction

In this paper we consider the following coupled flow and transport problem that arise in porous media:

$$\partial_t(\beta p) - \nabla \cdot (\mathbf{K}\nabla p) = q, \tag{1.1}$$

$$\partial_t(\phi c) + \nabla \cdot (c\mathbf{u} - \mathbf{D}\nabla c) = f. \tag{1.2}$$

Eq. (1.1), often referred to as the Darcy flow equation, governs conservation of mass for a slightly compressible single-phase fluid in a porous media. Here $p$ represents pressure and $\mathbf{u} = -\mathbf{K}\nabla p$ the Darcy velocity. The second equation (1.2) is known as the transport equation, and describes advective and diffusive transport of a concentration $c$. Such transport models are employed in modeling tracers in a porous media [1]. Choosing compatible numerical solvers for the flow and transport equations may be of importance for accuracy, stability and conservation properties [2]. Here we discuss using a continuous Galerkin (CG) finite element method for the flow equation and apply a postprocessing method to compute fluxes on element boundaries to obtain local conservation. A discontinuous Galerkin (DG) finite element method with upwinding is employed for the transport equation [3,4]. DG allows for discontinuities in the solution and has the advantages of local mass conservation, less numerical diffusion, favorable h- and p-refinement, handling of discontinuous coefficients, and efficient implementation.

CG is a well-developed numerical discretization for partial differential equations. It is numerically efficient for problems requiring dynamic grid adaptivity. It is known that CG requires postprocessing to obtain locally conservative fluxes on element boundaries [5–15]. This has been the topic also for studies of environmental modeling in bays and estuaries where CG has been employed for shallow water equations [16]. Applying non-conservative flux to the transport equation may result in non-physical concentration solutions [17,18,13].

Computing fluxes for CG models has been considered in many technical papers; we briefly describe some well known results and note that the list is incomplete. Optimal postprocessing of fluxes on element boundaries for one-dimensional problems was studied by Wheeler [19] and generalized by Dupont [20]. Douglas et al. [21] analyzed methods for approximating fluxes on the domain boundary for multi-dimensional problems based on the approach of J. Wheeler [22]. Postprocessing of locally conservative (or self-equilibrated) fluxes on element boundaries for multi-dimensional problems was studied by Ladeveze and Leguillon [23] for error estimation purposes. Ainsworth and Oden [5] proved the existence of such self-equilibrated fluxes for general CG methods including 1-irregular meshes with hanging nodes. Superconvergence of recovered gradients of linear CG approximations for elliptic and parabolic problems was treated by Wheeler and Whiteman [24,25].

For completeness we mention alternative schemes to CG for the pressure equation; mixed finite element methods [26], dual-grid and control volume methods [27], finite volume methods [28], mimetic finite difference methods [29], and DG [30]. All of these are conservative in the sense that they either are formulated in a mixed form so that locally conservative fluxes are obtained directly without the need for *any* postprocessing, or have an embedded local conservation statement in their derivation so that locally conservative fluxes can be calculated in a straightforward manner from the pressure solution. Recent papers [12,14] have observed that CG with postprocessing on the dual grid is more robust than standard control volume approaches. Here the postprocessing involves only local calculations. It is well known that for Laplace's equation, control volume and CG on the dual grid are equivalent. Lack or complexity of dynamic grid adaptivity is a disadvantage for many of the methods mentioned above. DG is promising both with respect to local conservation and dynamic grid adaptivity, but is computationally costly due to a high number of degrees of freedom. A conservative scheme based on enrichment of CG was proposed by [17] for elliptic problems and later extended to parabolic equations in [18].

The postprocessing method we propose in this paper is built upon the work of Sun and Wheeler [10] and Larson and Niklasson [9] for the steady-state flow model (Eq. (1.1) with $\beta = 0$). Both of these papers present an algorithm for computing conservative fluxes on element boundaries. Here a given general non-conservative flux approximation is modified by adding piecewise constant corrections which are minimized in a given norm. The minimization requirement ensures that the postprocessed flux has the same order of convergence as the original flux. The works by [10] and [9] have strong similarities and are in fact identical under some specific choice of parameters, but have been formulated differently. While a variational formulation is used in [9], the method is presented elementwise in [10]. In this paper we present both and demonstrate the relationship between the two results. We mention that these postprocessing methods have been applied in a series of recent works [31–34].