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On the behaviour of fully-discrete flux reconstruction schemes

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Highlights

- We investigate fully discrete flux reconstruction schemes in the context of ILES.
- Three high-order spatial discretizations are considered including FRDG and FRSD.
- Two explicit and two implicit high-order Runge–Kutta schemes are considered.
- We find strong dependence of dispersion/dissipation on choice of temporal scheme.
- Linear and non-linear experiments verify fully-discrete von Neumann analysis.

Abstract

In this study we employ von Neumann analyses to investigate the dispersion, dissipation, group velocity, and error properties of several fully-discrete flux reconstruction (FR) schemes. We consider three FR schemes paired with two explicit Runge–Kutta (ERK) schemes and two singly diagonally implicit RK (SDIRK) schemes. Key insights include the dependence of high-wavenumber numerical dissipation, relied upon for implicit large eddy simulation (ILES), on the choice of temporal scheme and time-step size. Also, the wavespeed characteristics of fully-discrete schemes and the relative dominance of temporal and spatial errors as a function of wavenumber and time-step size are investigated. Salient properties from the aforementioned theoretical analysis are then demonstrated in practice using linear advection test cases. Finally, a Burgers turbulence test case is used to demonstrate the importance of the temporal discretization when using FR schemes for ILES.

Keywords: Fully-discrete; High-order; von Neumann; Discontinuous Galerkin; Flux reconstruction; Runge–Kutta

1. Introduction

Unsteady scale-resolving computational fluid dynamics (CFD) simulations, such as large eddy simulation (LES) and direct numerical simulation (DNS) rely on two different types of discretization when using the method of lines. A spatial discretization is required to obtain the divergence of the flux functions in space, and a temporal discretization is required to advance the solution in time. Therefore, the numerical properties of particular space–time discretizations rely on the underlying properties of both the spatial and temporal schemes that are being used, and

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how they interact. Recent advances have led to the development of several high-order accurate unstructured spatial discretizations, which are suitable for unsteady flows around complex geometries. These include the discontinuous Galerkin (DG) [1–3] and spectral difference (SD) [4,5] methods, amongst others. Recently, the flux reconstruction (FR) approach was proposed by Huynh [6] as a unifying framework for high-order unstructured numerical methods. FR does not describe a single scheme, but is in fact a family of schemes. Huynh [6] described several linearly stable schemes, including one equivalent to a collocation nodal discontinuous Galerkin method, henceforth referred to as FRDG, and another equivalent to an energy-stable SD method, henceforth referred to as FRSD. High-order methods, such as the FR, DG, and SD approaches, are particularly appealing for simulations of complex unsteady flows [7–10]. They have been found to provide more accurate solutions with fewer degrees of freedom and reduced computational cost relative to industry-standard second-order schemes [11]. The FR approach has been shown to be particularly accurate for scale-resolving simulations of complex turbulent flows, including DNS and implicit LES (ILES) [7,11,8,9,12].

Unsteady simulations using the FR, DG, and SD approaches are typically performed with high-order accurate temporal discretizations. Explicit Runge–Kutta [13,14] (ERK) schemes are, perhaps, the most popular choice for problems that are numerically non-stiff. They are able to achieve high levels of accuracy with a low cost per time-step. However, ERK schemes are only conditionally stable [15], meaning the time-step size is limited. Problems with significant numerical stiffness often necessitate the use of A-stable and L-stable implicit schemes. Popular methods include the implicit Runge–Kutta and backward differentiation formula (BDF) schemes [15]. Since there are no A-stable BDF schemes with order greater than two [15], we are only interested in implicit Runge–Kutta schemes for the current study. Of the available implicit Runge–Kutta schemes, the singly diagonally implicit Runge–Kutta (SDIRK) methods are particularly appealing [15]. They require the solution of a relatively small system of equations at each stage, when compared to more general implicit Runge–Kutta methods. They also allow for a quasi-Newton approach to be employed for non-linear problems, whereby the same Jacobian matrix can be used for several stages within a time-step, or across several time-steps, reducing the number of costly preconditioning operations [7,9]. The most popular methods for scale-resolving simulations of turbulent flows appear to be third- and fourth-order ERK schemes and third-order SDIRK schemes [16,9,17,12]. We also consider a popular five-stage fourth-order SDIRK scheme in the current study [15].

Previous studies of the FR approach have investigated the behaviour of various energy-stable FR (ESFR) schemes using semi-discrete von Neumann analysis [6,18,19]. This type of analysis provides insights into the dispersion, dissipation, numerical error, and group velocity properties of a semi-discrete scheme in the absence of temporal errors. For example, Huynh [6] investigated the behaviour of semi-discrete FRDG, FRSD, and similar schemes using von Neumann analysis. Similarly, Vincent et al. [18] and Vermeire and Vincent [19] performed von Neumann analysis for families of provably-stable semi-discrete ESFR schemes. However, none of the aforementioned studies investigated the behaviour of fully-discrete FR schemes. Yang et al. [20] showed that fully-discrete DG schemes can exhibit significantly different behaviour from their semi-discrete forms. They provided analytical expressions for the dispersion and dissipation behaviour of several schemes, with emphasis on their behaviour in the asymptotic low-wavenumber limit. The current study will follow the approach of Yang et al. [20] to investigate the fully-discrete behaviour of FRDG, FRSD, and a particularly appealing ESFR scheme identified by Vermeire and Vincent [19]. For the current study we restrict our analysis to FR schemes of polynomial degree $k = 4$, which is a common choice for scale-resolving simulations of turbulent flows [9,17,12,21,22]. However, the procedure described herein is readily generalized to all polynomial degrees. It is expected that results from this study will inform the use of fully-discrete FR schemes for DNS and LES of unsteady turbulent flows. The behaviour of fully discrete schemes at the highest-resolved wavenumbers is of particular interest for ILES, since ILES relies on numerical dissipation at these wavenumbers to dissipate energy from the flow, acting as a subgrid scale model [9]. Therefore, while Yang et al. [20] were primarily interested in the behaviour of schemes in the asymptotic limit, we are interested in the behaviour of FR schemes for all resolvable wavenumbers and implications on future large-scale ILES simulations.

This paper is structured as follows. In Section 2 we present a formulation of the FR approach following Huynh [6]. We will also summarize the correction function formulation for polynomial degree $k = 4$ provided by Vincent et al. [23], which yields stable-symmetric-conservative ESFR correction functions. In Section 3 we will describe fully-discrete von Neumann analysis, which follows from the work of Yang et al. [20]. We will then present results for several popular FR schemes with ERK and SDIRK time-stepping. In Section 4 we will investigate the behaviour of these fully-discrete schemes via numerical experiments including advection of a sine wave, a marginally-resolved